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On the Several Kinds of Number in Bolzano

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B6 Bolzano's Mathematics and the General Methodology of the Sciences
CLMPST 16, Prague
8 August 2019

Mottos

it is most convenient to understand by the word 'number' only that which common speech does, and what mathematicians distinguish as the *whole* or real [*wirklich*] numbers. (*RZL* §2, p. 17).

Now of course the kindergarten-numbers [*Kleinkinder-Zahlen*] appear to have nothing whatever to do with geometry. But that is just a defect in the kindergarten-numbers. ... Counting, which arose psychologically out of the demands of business life, has led the learned astray. (Frege 1979, p. 277)

Contents

Desiderata for a Theory of Numbers

1. In as much as the laws of arithmetic are true, to explain the source of these laws.
2. In as much as arithmetical truths are a priori, to explain this apriority.
3. To explain the applicability of arithmetic.
4. To elucidate the sense of arithmetical propositions.
5. In as much as numbers are objects, to elucidate their nature as objects.
6. To recognise and explain the different kinds of number (natural, integer, rational, irrational, complex).

Frege’s Response

F1 Laws of arithmetic are logically–analytically true, on the basis of definitions and the laws of logic.

F2 Arithmetical truths are analytic and therefore a priori. Numerical propositions (like “Mars has two moons”) are (mainly) empirical and a posteriori.

F3 Numbers are so defined that their applicability is built into their definitions.

F4 Propositions of pure arithmetic are about objects called numbers; in propositions in which numerical expressions occur, these are second-order quantifier expressions.

F5 Numbers are the extensions of concepts under which the extensions of concepts of objects fall.

F6 Natural numbers are fundamentally different from real and complex numbers. Real numbers are the extensions of concepts expressing proportionality among relational magnitudes. Complex numbers are not treated by Frege.

Where Frege Went Wrong

1. His logic was inconsistent. (Other than that, Mrs. Lincoln …)
2. His account of numerical expressions (e.g. "There are two moons of Mars") is wrong.
    a. He treats expressions like 'there are two' and 'are two' as stating properties of *concepts*.
    b. But concepts like *moon of Mars* are single items (functions) and so 'are two' is falsely predicated of them. What *are* two *are* the moons of Mars.
3. By ignoring plural expressions, he overlooked the true bearers of number (cardinality) properties — *collections*. Others (Aristotle, Euclid, Bolzano, Mill, Weierstraß, Husserl, Russell (early, pre-Fregean), Sir Thomas Cobleigh …) got this *right*.

Bolzano on Collections

A **collection** (*Inbegriff*) is anything that is a whole with parts, or is composite.

BB thinks [compositeness] is probably indefinable, a simple idea.

Collections include both integrated individuals (this table) and pluralities (the people in this room now). No clear demarcation.

A collection where the relationships among the members is of no consequence is a **multitude** (*Menge*).

A collection consisting of several parts of the same kind is a **plurality** (*Vielheit*)

A collection, all of whose members are of kind **A**, is a **concrete plurality** of **A**s**.**

A [the] collection comprising all **As** is the **totality** (*Allheit*) of **A**s

A collection where the parts of the parts are parts of the whole is a **sum** (*Summe*).

A collection whose members are ordered so that each is preceded and succeeded by exactly one thing according to a single law is a **series** (*Reihe*).

Kinds of Number

Every individual of kind **A** is a **unit of A**

A concrete plurality of two **A**s is a **two of As**

Similarly, a three of As, a four of As, etc.

The attribute (*Beschaffenheit*) ***a*** in virtue of which something is an **A** is an **abstract unit** of kind **A**

An attribute in virtue of which something is a single something is an **abstract unit in general**.

Something may be a single thing in one respect but not in another (*WL* § 86)

(BB does not give an example. Here is a suggested one: these eleven people are a unit with respect to the idea [cricket team] but not with respect to the idea [human being].)

Number as a Member of a Series

Consider a series [*Reihe*] which is formed by taking a unit of arbitrary kind A as the first member and in which every further member is a sum which is derived by adding a new unit to a thing that is equal to the immediately preceding member. Every member of this series I wish to call a *number*, provided it is thought under an idea which indicates the way in which it was derived. One can easily see that any *finite* plurality can be represented by a number as far as its quantity is concerned, but that no number can be given for the *infinite* plurality, which is why we call it *uncountable*. (*WL* § 87)

Reine Zahlenlehre 1

If we construct a series whose first member is a unit of any kind **A**, and every other member is a sum which appears when we we take an object which is like the next previous member and connect it to a new unit of the kind **A**: I then call every member of this series a *number*, in so far as I can think of this member by an idea which gives us its manner of arising. To distinguish this from other series which appear when instead of things of kind **A** things of another kind of unit [occur] I call the members of the aforementioned series *numbers of kind* ***A***, or numbers for which the unit **A** *is the basis.* The attribute in virtue of which each of these members is a number (which is preserved when the objects themselves which are taken as units are exchanged) I call a *number* in the *abstract meaning* of the word, or an *abstract number*; and in contrast to such abstract numbers (i.e. the mere attributes) I call the members themselves *concrete numbers* or numbers in the *concrete* meaning of the word. In German, these concrete numbers, especially excepting the first or the unit, are also called *Anzahlen*. Finally I call the whole series the *number series*, or to distinguish it from other series whose members are also numbers, the *natural series of numbers*, or as some others do, *the series of natural numbers*. (*RZL* §1)

Natural Numbers

A number (whether abstract or concrete) is *always a member of a series*:

This concrete seven of **A**s (call the multitude **H**) is a number because it is seventh in a series of multitudes

This **A** (which is one of **H**)
This other **A** of **H** distinct from the previous one, and it
This other **A** of **H** distinct from the previous two, and them
This other **A** of **H** distinct from the previous three, and them
This other **A** of **H** distinct from the previous four, and them
This other **A** of **H** distinct from the previous five, and them
This other **A** of **H** distinct from the previous six, and them

(and there can be no more because all the **A**s in **H** are now included: the last multitude is just **H**)

Natural Numbers II

Problem: There are 7! = 5040 series of nested **A**-multitudes terminating in **H**

Bolzano skates over this problem:
"The attribute in virtue of which each of these members is a number (which is preserved when the objects themselves which are taken as units are exchanged) I call a *number* in the *abstract meaning* of the word"

He should have either
(a) Proved equivalence among and then abstracted from the different series terminating in **H**, or (preferably, in my view)
(b) Let the cardinality properties of multitudes order themselves by their nature (Frege and Whitehead–Russell)

(b) typically involves accepting *equinumerosity* (existence of a bijection) as the criterion for *same number*, something Bolzano rejects for infinite collections.

Exercise for the reader: Show (b) using instead a part–whole analysis of *same number* (cf. Trlifajová, this session; surely easier for the finite case)

Sequences and Series

A Bolzano series (*Reihe*) does not admit of repetitions: a sequence (in the modern sense) does. (A series is simply a linearly ordered multitude.)

Bolzano considers series with repetitions,
e.g. $1^0, 1^1, 1^2, 1^3, 1^4, \ldots$ and $i, 2i, 4i, 8i, \ldots$ – and **rejects** them:

"it is easy to explain how we may speak of a sequence when several or even all of the objects corresponding to our ideas are equal, once we remark that in such cases it is not the objects themselves, but our ideas of them that are envisaged as terms of the sequence. These ideas can, understandably, be different from one another, and each can be derived from its predecessor according to a rule that is valid for all, without it being the case that the objects of these ideas are different. Indeed, the ideas may have no objects at all. The ideas $1^0, 1^1, 1^2, 1^3, \ldots$ are easily distinguished even though they all have one and the same object, namely, unity, and although they can all be set = 1. So too the ideas $\sqrt{-1}$, $2\sqrt{-1}$, $4\sqrt{-1}$ are easily distinguished, even though none of them has any object."
(*WL* § 85)

Sequences and Series II

A sequence is nowadays usually defined as a function from the natural numbers to members of some non-empty set, but this presupposes numbers. Is there a way to simulate sequences using Bolzano's methods? Yes. Here are two:

1. Imitating the modern definition:

Replace the natural numbers as function domain by this omega-sequence:
e, the proposition that something exists
T(*e*), the (distinct) proposition that *e* is true
T(T(*e*)), the further distinct proposition that T(*e*) is true … and so on.

2. Keeping closer to Bolzano's idea of using collections:

For any finite or denumerably infinite multitude *M* consider, for each member *x* of *M,* the following infinite ordered collection of *clones* of *x*
The first is *x* itself. The second is the proposition [E!*x*] that *x* exists. Call this *x*´.
The third is the proposition that *x*´ exists: [E![E!*x*]]: call this *x*´´. It is distinct from *x* and *x*´. Iterate.

Sequences and Series III

Now consider the following ordered set O. It has the following characteristics:

(1) Every member of M occurs exactly once in O.
(2) Some member a of M is the first member of O.
(3) Every subsequent member of O is
either (a) a member b of M (distinct from a, obviously)
or (b) a clone c of a member b of M such that
(i) b occurs in O before c
(ii) c is the successor clone of b to the highest clone of b occurring in O before c.

O then represents the sequence with possible repetitions of the members of M such that a is the first member, each member of M occurs in O where its first occurrence of the sequence is, and for each member x of M that x occurs in the sequence more than once, its (n+1)st occurrence is represented in O by the nth clone of x.

Ordered from Unordered Multitudes

We may even represent ordered multitudes by unordered ones, provided we allow multitudes of multitudes. An ordered multitude *abcd…k….* may be adequately represented by the higher-order multitude *a ab abc abcd abcd…k* etc. (where in each member and among the members the order does not matter). In this way sequences may also be represented by unordered multitudes, provided we allow multitudes of higher than first order.

Whether BB would have approved is unclear.

Bolzano and the Desiderata

Bolzano does not to my knowledge attempt an axiomatic or logical derivation of the laws of arithmetic. I don't know what he thinks about their apriority.

A proposition like [2 + 2 = 4] is not Bolzano-logically-analytic unless all the ideas in it are logical. (Replace [4] by [5], or [+] by [–])

The proposition [$2n + 2n = 4n$] is true and analytic with respect to the idea n.

By distinguishing concrete and abstract collections of given cardinalities Bolzano easily copes with applications.

The sense of arithmetical expressions, both pure and applied, is adequately explained.

Numbers as objects are members of nested series. Fair explanation but could be improved.

The "higher numbers" are dealt with by Bolzano in the theory of quantity (at length).

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Děkuji
Thank you
Go raibh maith agat