# Interpreting Bolzano's RZ VII §107

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# Aims

By looking at Bolzano's own mathematical practice, we aim to show:

1. That one of Bolzano's most famous contributions to mathematics might deserve a more charitable interpretation than previously thought
2. This follows once we accept that Bolzano's reasoning is not strictly set theoretic
3. That we can solve some open interpretive issues regarding Bolzano's measurable numbers

# Main claim

Bolzano's own foundational concepts allow for a sound interpretation of one of his most interesting contributions to analysis.

# Overview

1. Introduction
2. Measurable numbers
3. RZ VII §107
   - Archimedeanity for measurable numbers
   - Parts and wholes, sets and subsets
4. Conclusion

# Measurable Numbers

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Measurable numbers are a particular kind of infinite number expressions, or infinite number concepts, that can be ‘measured’.

# *Measurable* Numbers

Any number expression $S$ is measured or determined by approximation iff for any positive integer $q$ one can find an integer $p$ and 'strictly positive number expressions' (so, without '-') $P^1, P^2$ such that
$\frac{p}{q} + P^1 = S = \frac{p+1}{q} - P^2$.

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The measuring is analogous to working out the decimal representation of (irrational) numbers, in the following sense: we can measure $\sqrt{2}$ in terms of e.g. fractions of 10 (then $\sqrt{2}$ is between $\frac{14}{10}$ and $\frac{15}{10}$ ) or fractions of 100, (then $\sqrt{2}$ is between $\frac{141}{100}$ and $\frac{142}{100}$), etc.

# RZ VII §107

*Suppose the infinitely many measurable numbers $X^1, X^2, X^3, \ldots, X^n \ldots, X^{n+r}, \ldots,$ which we can consider as the terms of an infinitely continuing series distinguished by the indices $1, 2, 3, \ldots, n, \ldots, n+r, \ldots,$ proceed according to such a rule that the difference between the nth term and the $(n+r)$ th term of the series, i.e. $(X^{n+r} - X^n)$, considered in its absolute value, always remains, however large the number n is taken, smaller than a certain fraction $\frac{1}{N}$ which itself can become as small as we please, providing the number n has first been taken large enough. Then I claim that there is always one and only one single measurable number A, of which it can be said that the terms of our series approach it indefinitely, i.e. that the difference $A - X^n$ or $A - X^{n+r}$ decreases indefinitely in its absolute value merely through the increase of n or r.[2]*

# Cauchy's criterion

In modern terms, this is a statement of the sufficiency of Cauchy's criterion for the convergence of a 'series'.
A series (sequence) is a Cauchy series if and only if: for all $\epsilon$ there is $k$ such that for all $m > n > k$ $|x_m - x_n| < \epsilon$.

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**Sufficiency of Cauchy's criterion for convergence**

If a sequence is Cauchy, then it converges.

The theorem of §107 and the sufficiency of Cauchy's criterion are equivalent only if Bolzano's measurable numbers obey Archimedeanity (that is, only if Bolzano's measurable numbers do not include infinitesimals)

# The proof

Step 1 Assume the series is nondecreasing (similar proof if nonincreasing). By assumption it follows that for any $q$, $|X^{n+r} - X^n| < \frac{1}{q}$ (if $n$, $r$ large enough). If $X^n = \frac{p}{q}$ for some $p$, that surely $X^n$ and all following $X^{n+r}$ approach $\frac{p}{q} + P^1 = \frac{p+1}{q} - P^2$ indefinitely.

Step 2 Suppose $X^n \neq \frac{p}{q}$. Measurability implies $\frac{\pi}{q} < X^n < \frac{\pi+1}{q}$. Then either all $X^{n+r} < \frac{\pi+1}{q}$, or not. If the former, then either the difference $\frac{\pi+1}{q} - X^{n+r} > X$ for some $X$, all $r$ large enough, or not. Then if it is, $A = \frac{\pi}{q} + P^3 = \frac{\pi+1}{q} - P^4$; if it isn't, then the difference approaches $\frac{\pi+1}{q}$, which is our $A$. If not all $X^{n+r} < \frac{\pi+1}{q}$, there are some that are bigger. Then $A = \frac{\pi+1}{q} + P^5$, but also $X^{n+r} < X^n + \frac{1}{q} < \frac{\pi+1}{q} + \frac{1}{q}$, so $A = \frac{\pi+2}{q} - P^6$. **Bottom line: Bolzano gives us a procedure of how to construct an adequate $A$ in any case**.

# The proof

Step 3 Uniqueness: if the series converges to both $A \neq B$, then $A - X^n = \omega_1$ and $B - X^n = \omega_2$ (where the $\omega$'s are infinitely small numbers). But then $A - B = \omega_1 - \omega_2$, which (according to the amended definition of equality for measurable numbers) is impossible. So $A = B$.

Step 3 requires that all infinitesimals (if that is what the $\omega$'s are) are equal (equivalent) to 0, so in the measurable numbers system there are no infinitesimal numbers, as they collapse to zero. Therefore, for step 3 to go through, the measurable numbers have to obey the Archimedean property: for any two measurable numbers $x, y, x < y$, there is a natural number $n$ such that $xn > y$. If $x$ were an infinitesimal, this property would fail.

# The proof

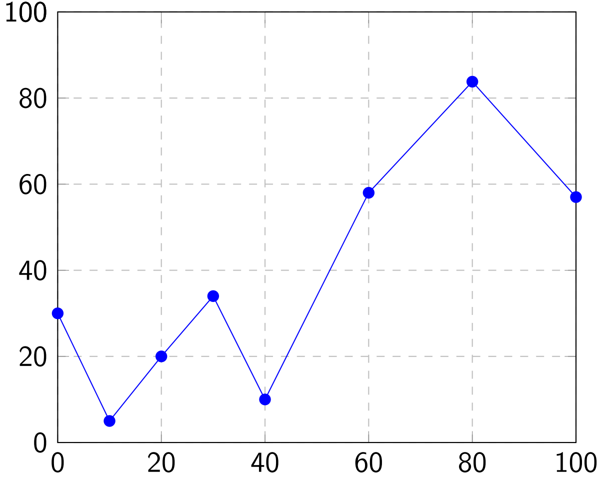
Step 4 Extend the existence proof of $A$ for when the series is not monotone. First, argue that any series contains a Cauchy subseries. Then, by steps 1-3, this subseries converges to an A. Second, argue that the entire series converges to the same A as the subseries.

# Subsets, subset selection, and (sub)sequences

Here we focus on the argument for the existence of a Cauchy subseries. Consider how Step 4 begins:

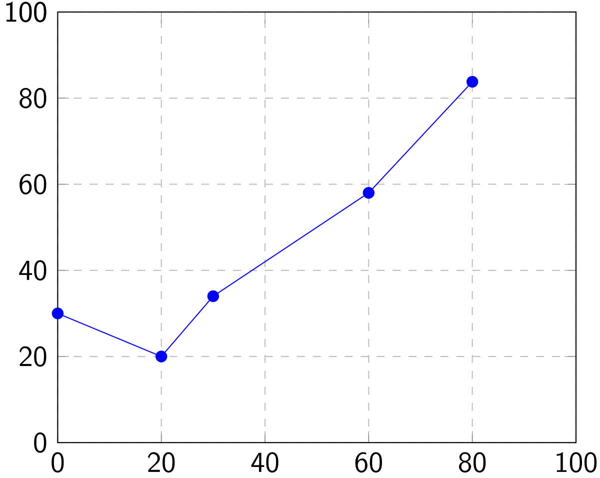
> *if the series of numbers $X^1, X^2, X^3 \ldots$ is of such a kind that for every nth term of it, $X^n$, there are, among the succeeding terms, not only some that are greater but also some that are smaller, then we can* **select** *from the whole collection of these numbers only that part of it which has the property that every succeeding term is greater or smaller than the preceding term.*

# Bolzano's construction of the subsequence



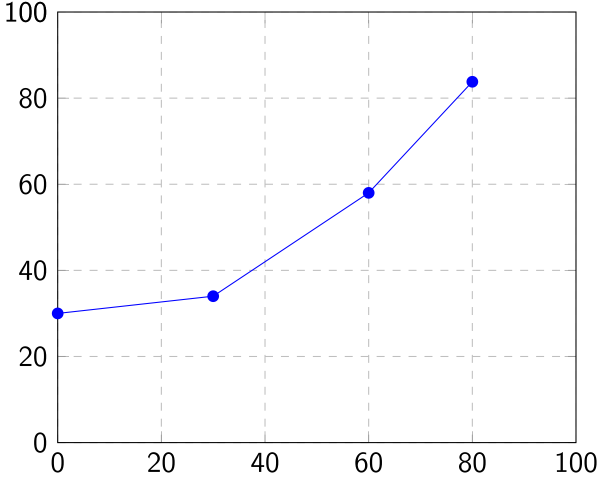
# Bolzano's construction of the subsequence

If we interpret Bolzano as appealing to subset selection:



# Bolzano's construction of the subsequence

The more charitable interpretation:



# A rule for a series, a series for a rule

- At first glance, it is hard to understand what Bolzano's argument is
- We have just given one possible explanation
- Advantage: no more 'gap' in Bolzano's proof; gain in understanding of his reasoning
- Moreover: Precise sense in which Bolzano's reasoning is not set theoretic (the 'property of the part' he talks about cannot be reduced/expressed in terms of property of individual elements, so we cannot use subset selection).

# Take-away message

By looking at Bolzano's mathematical text closely, we were able to conclude the following:

1. Bolzano's reasoning is best expressed in terms of parts and wholes, not easily reduced to set theoretic reasoning
2. This reading is more charitable towards Bolzano's proof of the Bolzano-Weierstrass theorem
3. The proof gives evidence for the no-infinitesimal reading of RZ VII

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Thank you!