Bolzano and the Part-whole Principle

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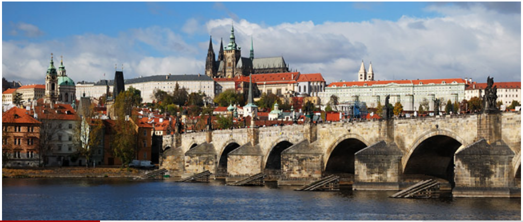

Two great questions

1. The existence of actual infinity?
2. How to measure infinities?

- Bernard Bolzano: *Paradoxes of the Infinite*, 1848
- Georg Cantor: *Contributions to the Founding of the Theory of Transfinite Numbers*, 1883

Galileo's paradox

G. Galilei, *Discorsi e dimostrazioni matematiche*,1638

1	2	3	4	5	6	7	8	9	10	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	...
1	4	9	16	25	36	49	64	81	100	

- Part-whole principle. The whole is greater than its part.
- Cantor's principle. One-to-one correspondence

How do we obtain the squares?

- We select them from numbers.
- We create them from numbers.
  „Every number is a root of some square." (G. Galilei)

Bolzano and one-to-one correspondence

- *Let us turn now to the consideration of a highly remarkable pecularity, which can occur, indeed actually always occurs, in the relationship between two mutitudes if they are both infinite, . . . Merely from this circumstances we can in no way conclude that these two multitudes are equal to one another.* §20.
- *An equality of these multiplicities may only be concluded if some other reason is added, such as that both sets have one and the same determining ground [Bestimmungsgrűnde]. e.g. they have exactly the same way of being formed [Entstehungsweise].* §21.

Bolzano's infinite series

1. $P\ = 1 + 2 + 3 +\ 4 + \cdots +$ in inf.
2. $S\ = 1 + 4 + 9 + 16 + \cdots +$ in inf.
3. $N_0 = 1 + 1 + 1 + 1 + \cdots +$ in inf.
4. $N_m = \underbrace{\ldots}_{m} 1 + 1 + 1 + \cdots +$ in inf., the first m terms are omitted.

- *The multitude of terms in the series P, S is certainly the same.*

$$S > P$$

- *If we subtract P from S m-times we obtain* $S - mP > 0$. Thus

$$S >> P$$

- *We obtain the certain and quite unobjectionable equation.*

$$n = N_0 - N_n.$$

Interpretation

Bolzano's series $\sim$ sequences of partial sums

Definition 1.

Let $a_1, a_2, \dots a_n, \dots \in \mathbb{N}$, $s_n = a_1 + \dots + a_n$. We interpret the series

$$a_1 + a_2 + a_3 + \dots + \text{in inf.} \sim (s_1, s_2, s_3, \dots) = (s_n)$$

Bolzano's natural number series $\sim \{(s_n)_{n \in \mathbb{N}}, s_n \in \mathbb{N} \wedge s_n \leq s_{n+1}\} = S$

1. $P = 1 + 2 + 3 + \ 4 + \dots \text{in inf.} \sim (1, 3, 6, 10, \dots) = (\frac{n \cdot (n+1)}{2})$
2. $S = \ 1 + 4 + 9 + 16 + \dots \text{in inf.} \sim (1, 5, 16, 32, \dots) = (\frac{n(n+1)(2n+1)}{6})$
3. $N_0 = 1 + 1 + 1 + 1 + \dots \text{in inf.} \sim (1, 2, 3, \dots) = (n)$
4. $N_m = \underbrace{\dots}_{m} 1 + 1 + 1 + \dots \text{in inf.} \sim (\underbrace{0 \dots 0}_{m}, 1, 2, 3, \dots)$

Equality and order

Bolzano's condition of associativity and commutativity of terms of series

$$(A + B) + C = A + (B + C) = (A + C) + B.$$

Definition. Let $(a_n), (b_n)$ be two sequences of natural numbers

- $(a_n) =_{\mathcal{F}} (b_n)$ if and only if $(\exists m)(\forall n)(n > m \Rightarrow a_n = b_n)$.
- $(a_n) <_{\mathcal{F}} (b_n)$ if and only if $(\exists m)(\forall n)(n > m \Rightarrow a_n < b_n)$.

Fréchet filter $\mathcal{F}$ on $\mathbb{N}$.

- $(a_n) + (b_n) = (a_n + b_n)$.
- $(a_n) \cdot (b_n) = (a_n \cdot b_n)$.

Theorem 1.

- Let S be the set of non-decreasing sequences of natural numbers. Then $(S, +, \cdot, =_{\mathcal{F}}, <_{\mathcal{F}})$ is a partial ordered non-Archimedean ring.
- All Bolzano's propositions are valid in this interpretation.

Set-sizes

Definition 2.

A set A is *computable* if it can be arranged as a disjoint union of finite parts, i.e. subsets of A.

$$A = \bigcup \{A_n, n \in \mathbb{N}\}.$$

- The *set-size* of A is expressed by the Bolzano series

$$|A_1| + |A_2| + |A_3| + \ldots \text{in inf.}$$

- The *characteristic sequence* $\chi(A)$

$$\chi(A) = (|A_1|, |A_2|, |A_3| \ldots)$$

- The *size sequence* $\sigma(A)$

$$\sigma(A) = (|A_1|, |A_1| + |A_2|, |A_1| + |A_2| + |A_3|, \ldots)$$

Natural numbers and their subsets

A canonical arrangement of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$

$$\mathbb{N}_n = \{n\}$$

1. $\chi(\{3,4\}) = (0,0,1,1,0,0\dots)$, $\sigma(\{3,4\}) = (0,0,1,2,2,2,\dots) =_{\mathcal{F}} 2$.
2. $\chi(\mathbb{N}) = (1,1,1,1,1,\dots)$, $\sigma(\mathbb{N}) = (1,2,3,4,5,\dots) = \alpha$.
3. $\chi(\mathbb{N} \setminus \{3,4\}) = (1,1,0,0,1,1,\dots)$,
   $\sigma(\mathbb{N} \setminus \{3,4\}) = (1,2,2,2,3,4,\dots) =_{\mathcal{F}} \alpha - 2$.
4. S - squares, $S = \{1,4,9,16,\dots\}$, $\chi(S) = (1,0,0,1,0,0,0,0,1,0\dots)$,
   $\sigma(S) = (1,1,1,2,2,2,2,2,3,3\dots) <_{\mathcal{F}} \alpha$.
5. E - even n., $\chi(E) = (0,1,0,1,0,1,\dots)$, $\sigma(E) = (0,1,1,2,2,3,\dots)$
6. O - odd n., $\chi(O) = (1,0,1,0,1,0\dots)$, $\sigma(O) = (1,1,2,2,3,4\dots)$
7. $\sigma(E) + \sigma(O) = \alpha$, $\sigma(E) < \sigma(O) + 1$.

Primes, squares, k-multiples

1. The set-size of primes $\sigma(P) = (\pi(n))$.

$$\frac{n}{\log n} \leq \pi(n) \leq \frac{3n}{\log n}$$

2. The set-size of squares $\sigma(S) = (1, 1, 1, 2, 2, 2, 2, 2, 3, 3 \ldots)$.

$$\sqrt{n-1} \leq \sigma_n(S) \leq \sqrt{n}$$

3. The set-size of k-multiples $\sigma(M_k) = (\underbrace{0, \ldots 0}_{k-1}, \underbrace{1, \ldots 1}_{k}, \underbrace{2 \ldots 2}_{k}, \ldots \ldots)$.

$$\frac{n-1}{k} \leq \sigma_n(M_k) \leq \frac{n}{k}.$$

- $\sqrt{n} < \frac{n}{\log n} \Rightarrow \sigma(S) <_{\mathcal{F}} \sigma(P)$.
- $(\forall k)(\exists m)(\forall n > m)(\frac{3n}{\log n} < \frac{n-1}{k} < \sigma_n(M_k)) \Rightarrow \sigma(P) <_{\mathcal{F}} \sigma(M_k)$

Canonical arrangement

We extend the *canonical arrangement* gradually to be unique and to depend on the determination ground.

Theorem 2.

If A, B are canonically arranged computable sets then

$$\sigma(A \cup B) = \sigma(A) + \sigma(B) - \sigma(A \cap B)$$

Consequences:

1. $|A| = n$ is finite if and only if $\sigma(A) =_{\mathcal{F}} n$.
2. If A, B are disjoint then $\sigma(A \cup B) = \sigma(A) + \sigma(B)$.
3. If $A \subset B$ then $\sigma(A) <_{\mathcal{F}} \sigma(B)$. Part-Whole principle.

Integers

- Negative numbers $\mathbb{N}^-$ have the same canonical arrangement as $\mathbb{N}$.

  The set-size

  $$\sigma(\mathbb{N}^-) = \sigma(\mathbb{N}) = (1, 2, 3, 4, \dots)$$

- Integers $\mathbb{Z} = \mathbb{N} \cup \mathbb{N}^- \cup \{0\}$.

  The set-size

  $$\sigma(\mathbb{Z}) = \sigma(\mathbb{N}) + \sigma(\mathbb{N}^-) + \sigma(\{0\}) = (3, 5, 7, 9, 11, \dots) = 2\alpha + 1.$$

Cartesian product

Definition 3.

A *canonical arrangement* of a Cartesian product $A \times B$ of two canonically arranged sets $A = \bigcup\{A_n, n \in \mathbb{N}\}, B = \bigcup\{B_n, n \in \mathbb{N}\}$ is

$$(A \times B)_n = \bigcup\{A_i \times B_j, n = \max\{i, j\}\}.$$

	A_1	A_2	A_3	A_4	...
B_1	$A_1 \times B_1$	$A_2 \times B_1$	$A_3 \times B_1$	$A_4 \times B_1$	...
B_2	$A_1 \times B_2$	$A_2 \times B_2$	$A_3 \times B_2$	$A_4 \times B_2$	...
B_3	$A_1 \times B_3$	$A_2 \times B_3$	$A_3 \times B_3$	$A_3 \times B_3$	...
B_4	$A_1 \times B_4$	$A_2 \times B_4$	$A_3 \times B_4$	$A_4 \times B_4$	...
...	...	...	...	...	...

$$\sigma(A \times B) = \sigma(A) \cdot \sigma(B)$$

Rational numbers - the interval $I = (0,1]_{\mathbb{Q}} \subseteq \mathbb{Q}$

$I \sim \{[m,n] \in \mathbb{N} \times \mathbb{N};\, m, n \text{ are coprime and } m < n\} \cup [1,1]\}$

	1	2	3	4	5	6	7	8	9	...
1	1	0	0	0	0	0	0	0	0	...
2	1	0	0	0	0	0	0	0	0	
3	1	1	0	0	0	0	0	0	0	
4	1	0	1	0	0	0	0	0	0	
5	1	1	1	1	0	0	0	0	0	
6	1	0	0	0	1	0	0	0	0	
7	1	1	1	1	1	1	0	0	0	
8	1	0	1	0	1	0	1	0	0	
9	1	1	0	1	1	0	1	1	0	
...										...

- $\chi(I) = (1,1,2,2,4,2,6,4,6,\dots)$. Euler function
- $\sigma(I) = (1,2,4,6,10,12,18,22,28,\dots) < \frac{\alpha^2-\alpha}{2}$.

Rational numbers $\mathbb{Q}$

Positive rational numbers Q^+

$$\mathbb{Q}^+ \sim \mathbb{N}_0 \times I$$

- The set-size $\sigma(\mathbb{Q}^+) = \sigma(\mathbb{N}_0) \cdot \sigma(I) = (2, 6, 16, 30, 60, 84, 144, 198, 280, \dots) < (\alpha + 1) \cdot \frac{\alpha^2 - \alpha}{2} = \frac{\alpha^3 - \alpha}{2}$.

Rational numbers

$$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$$

- The set-size $\sigma(\mathbb{Q}) = 2\sigma(\mathbb{Q}^+) + 1 = (5, 13, 33, 61, 121, 169, 289, \dots) < \alpha^3 - \alpha$

Theory of numerosity

Benci, Vieri, Di Nasso, Mauro, *Numerosities of Labelled Sets: a New Way of Counting*, *Advances in Mathematics* 173, p. 50-67. 2003.
Benci, Vieri, Di Nasso, Mauro, *How to Measure the Infinite, Mathematics with Infinite and Infinitesimal Numbers*, World Scientific, 2019.

- The basic notions are defined, not justified.
- Some assumption are arbitrary, $\sigma((0,1]_{\mathbb{Q}}) = \alpha$.
- An ultrafilter. A linear odering of set-sizes. Some results are arbitrary,
- A basis of α-calculus, non-standard analysis.

- Founded on Bolzano's theory.
- A canonical arrangement is based on the determination ground.
- The Fréchet filter. Partial ordering of set-sizes. Unique results.
- More intuitive and simple.