On Bolzano's early rejection of infinitesimals

Elías Fuentes Guillén
UNAM

Introduction

1804: "I could never be satisfied with a completely strict proof if it [...] makes use of a fortuitous, alien, *intermediate concept*". (V.)

1810: "The chapter on irrational and imaginary quantities is even more unsettled, and in some parts full of self-contradictions. I do not want to mention anything here about the deficiencies that higher algebra [and] the differential and integral calculus have."

- "But if it should be decided one day that the *infinite*, or the *differential*, are nothing but *symbolic expressions* just like $\sqrt{-1}$ and suchlike [expressions], and likewise if it turns out that the method of proving truths [using them] [...] is always correct and logically admissible [...]" (§17)

Apropos the binomial theorem

"[The] assumption of an infinite series [...] is the assumption of the sum of infinitely many quantities, [and] every attempted calculation of its value is therefore an attempted calculation of the infinite, a true *calculus infinitesimalis*." (Vorrede)

- Quantities "that can become smaller than any given one."

"[S]hakiest foundations", e.g. the self-contradictory concepts of infinitely small quantities and the assumption that even zeros have a ratio to one another: "a nonentity has no properties."

$$\overset{(1)}{\omega} + \overset{(2)}{\omega} + \omega + \cdots + \overset{(m)}{\omega} = \Omega \quad (\S14\text{-}15)$$

He uses the notion of ω in his *Rein analytischer Beweis*.

The title of *Die drey Probleme* explicitly points out these are "solved without consideration of the infinitely small."

"[Some geometers] claim that in regard to the length [the difference between the piece of arc and its chord] is an infinitely small [quantity] of second order, while the difference between the arc and another drawn straight line will amount an infinitely small [quantity] of first order. Now where is the proof of these claims? Or is one supposed to regard judgements which in so many cases are composite as genuine basic truths?" (Vorrede)

Apropos Schultz

"Über das Fundament der Differentialrechnung" (1803) (5/2):

- The object of analysis are not quantities of a certain kind (E.): "not all variable quantities vary in time" (6/2; cf. 7/2).

- "Logically correct" definition of an infinitesimal as "< any alleged quantity, without already determining whether [it] is something real or nothing" (§25): "etwas Unmögliches soll man nicht als Begriff aufstellen".

- "Kästner distinguishes between a vanishing and an already vanished quantity: the former is the so-called infinitely small, which is not < any alleged quantity but can become smaller. Allein durch diese *subtilen Distinctionen* wird nichts gewonnen." (§32)

Apropos Carnot

Betrachtungen über die Theorie der Infinitesimalrechnung (1800; translation by Friedrich Hauff) (6/2):

- "Schlechterdings verwerflich" use of language: 0 is not a quantity and so it cannot be conceived as a limit. (23)

- Evanescents: "what great geometers thought they could express by saying that [they] were quantities considered not before they vanished, not after they vanished, but at the exact instant in which they vanished." (42)
  - "Êtres de raison": "Producte des Unverstandes". (41/6)

- Carnot's procedure ("une compensation d'erreurs") "[is] not scientific [and] by no means [gives] the objective ground of truth." (9)

Underlying problems

"[I]n [the law of continuity] actually lies the key for the resolution of the whole mystery of infinitesimal calculus." (6/2; Lacroix, 1797)

Bolzano's notion of ω hints at a notion of infinity different from the modern, actualist one?

- "Philosophizing mathematicians" have recovered "the well-founded distinction" between actual and potential infinite made by "ancient metaphysicians", deriving from it "the right consequence that only potential infinity may belong to the quantities." (Hauff, 1800, Zusäze)

"I still cannot get things straightened out with the concepts [of] 0, $\sqrt{-1}$ and ∞." (7/1)

The first version of ω

"Smaller [...] than any alleged quantity is a self-contradictory concept.

In contrast, the assertion that a certain quantity can be assumed smaller than any given [quantity] does not contain the slightest contradiction.

[T]his assertion should not be understood as if one and the same quantity, the very same (unmodified object), can be smaller than any given [quantity], but simply that [...] one can assume an even smaller [quantity] for each already assumed.

The former is the usual definition of the infinitely small, and is therefore to be rejected. Der zweyte Begriff ist sehr reell, und wirklich nöthig." (7/1)

Bolzano's later stance

"Likewise, I do not doubt that ∞ large quantities may be accepted. But that ∞ small [quantities] may also be accepted [is something that] remains doubtful to me. Nevertheless, I am very inclined towards the affirmative opinion." (2B.9/2; late 1814)

"The question is whether mathematicians, when they usually call number what the signs $\frac{1}{4}, \sqrt{2}, \sqrt{-1}, \infty, \frac{1}{\infty}, 0$ designate, speak [...] in a borrowed or real sense." (2A.8; 1830s)

"Arbitrary proposition: For abbreviation, we shall sometimes denote by the signs $\omega, \Omega, \omega^1, \omega^2,$ etc. numbers that can decrease indefinitely." (2A.8; 1830s)

Final remarks

Bolzano's early notions account for the tensions that existed between his new insights and views that he inherited.

- Process of refinement and increasing abstraction of the idea of quantity, which led to the development of a theory of real numbers.

- Dichotomies be/can become and alleged/given seem to emphasize ω's character as potentially infinitely small.

- Modern approach: abandonment of the core notion of variable quantity; embracement of a purely mathematical domain of objects, i.e. numbers, which was required to be a given continuous –as well as dense and ordered– one, and no longer –explicitly or implicitly– dynamic.

Thanks!