BMMS: On the several kinds of number in Bolzano

Modern philosophy of arithmetic almost invariably begins with a discussion of Gottlob Frege and his failed attempt to give a logicist derivation of the laws of arithmetic. A much less well known but similarly revolutionary treatment of the natural numbers and their close cousins was given half a century earlier, by Bernard Bolzano. Despite sharing many platonist assumptions with Frege, Bolzano's treatment of numbers is markedly more differentiated than that of Frege. It is therefore instructive to pull together Bolzano's somewhat scattered remarks (found principally in [1] § 85 ff., [2], [3] and [4]), investigate the variety of objects he was prepared to call "number", and see to what extent we can learn from this variety even today. We confine attention to what Bolzano himself called "the whole numbers". Those quantities which call for negative, rational and irrational numbers form in Bolzano a topic too vast for brief discussion. By contrast with his account of continuous magnitudes, his theory of numbers stands up remarkably well.

Bolzano's treatment of the numbers presupposes his theory of collections (*Inbegriffe*), itself a rich and extensive topic in his ontology. The four most important kinds of collection for Bolzano's theory of number are multitudes (*Mengen*), sequences (*Reihen*), sums (*Summen*) and pluralities (*Vielheiten*), whose informal definitions we give. On this basis, Bolzano distinguishes concrete and abstract units of a given kind *A*, concrete and abstract pluralities of *A*s, named and unnamed pluralities, and finally, the abstract natural numbers themselves. All of these distinctions make perfect sense and have straightforward application. The main flaw in Bolzano's treatment is his account of sequences, which, because he does not admit repetitions (unlike the modern concept of a sequence) undercuts his theory of counting. Sequences are usually now defined as functions from the natural numbers or an initial segment thereof, but that presupposes numbers. It would be preferable to avoid such dependence. We show how Bolzano's theory can be modestly extended and given a better basis by admitting collections of collections. The chief remaining gaps are then a proper understanding of the ancestral of a relation, and a recognition of collections of different infinite cardinalities: both of these innovations followed a good fifty years later.

Literature

[1] Bolzano, B. *Wissenschaftslehre*. Kritische Neuausgabe der §§ 46–90, ed. J. Berg. *Bernard Bolzano Gesamtausgabe* I/11/2. Stuttgart-Bad Cannstatt: Frommann–Holzboog, 1987. Translation in: *Theory of Science*, tr. P. Rusnock and R. George. Vol. 1. Oxford: Oxford University Press, 2014.

[2] Bolzano, B. *Einleitung zur Größenlehre: Erste Begriffe der allgemeinen Größenlehre,* ed. J. Berg. *Bernard Bolzano Gesamtausgabe* IIA/7. Stuttgart-Bad Cannstatt: Frommann–Holzboog, 1975.

[3] Bolzano, B. *Reine Zahlenlehre*, ed. J. Berg. *Bernard Bolzano Gesamtausgabe* IIA/8. Stuttgart-Bad Cannstatt: Frommann–Holzboog, 1976.

[4] *Paradoxien des Unendlichen*, ed. C. Tapp. Hamburg: Meiner, 2012.