Bernard Bolzano and the Part-Whole Principle

The embracing of actual infinity in mathematics leads naturally to the question of comparing the sizes of infinite collections. The basic dilemma is that Cantor's Principle (CP), according to which two sets have the same size if there is a one-to-one correspondence between their elements, and the Part-Whole Principle (PW), according to which the whole is greater than its part, are inconsistent for infinite collections [2].

Contemporary axiomatic set-theoretic systems, for instance ZFC, are based on CP. PW is not valid for infinite sets. Bernard Bolzano's approach primarily described in his *Paradoxes of the Infinite* from 1848 [4] relies on PW.

Bolzano's theory of infinite quantities is based on infinite series of numbers. PW leads to a special way of their treatment. They can be added, multiplied and sometimes we can determine their relationship. If we interpret infinite series as sequences of partial sums and factorize them by the Fréchet filter, then all properties determined by Bolzano will hold [6]. We obtain thus a partially ordered commutative non-Archimedean ring of finite, infinitely small and infinitely great quantities, where we can introduce the so-called „cheap non-standard analysis" [5].

The size of collections with regards to the multitude of their elements is another topic. Bolzano rejects CP as a sufficient criterion for equality of infinite multitudes. Further conditions are necessary; Bolzano refers to the need for having the same „determining ground". As to natural numbers Bolzano does not determine explicitly the relationship between their multitude and the multitudes of their subsets. Nevertheless, it is evident how to express these multitudes with help of Bolzano's infinite quantities. This is related to the Bolzano's special notion of a sum [3]. Similarly, infinite series express consistently multitudes of a union, of an intersection, and of a Cartesian product of natural numbers and their subsets. This extended conception of Bolzano is similar in its results to the theory of numerosities [1].

In *Paradoxes of the Infinite* Bolzano also investigates relationships among multitudes of points of segments, lines, planes and spaces. In this case the same "determining ground" means for Bolzano the existence of a one-to-one correspondence which is simultaneously an isometry. Also this part of Bolzano's work could be interpreted in the theory of numerosities.

References

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