On continuity
in Bolzano's 1817 *Rein analytischer Beweis*

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1. Ancient Greek characteristic of continuity was equally applied to space, time and motion, and was encapsulated in the statement: *continuous thing is divisible into parts that are infinitely divisible* (see [1]). In modern science, space and time are represented by real numbers, assuming $\mathbb{R}^n$ models space and $(\mathbb{R}, <)$ models time, while motion is represented by function. Accordingly, continuity splits into continuous order (continuity of real numbers) and continuous function.

We argue that this split of the meaning of continuity started in [3] and resulted from a duality of geometric line and function.

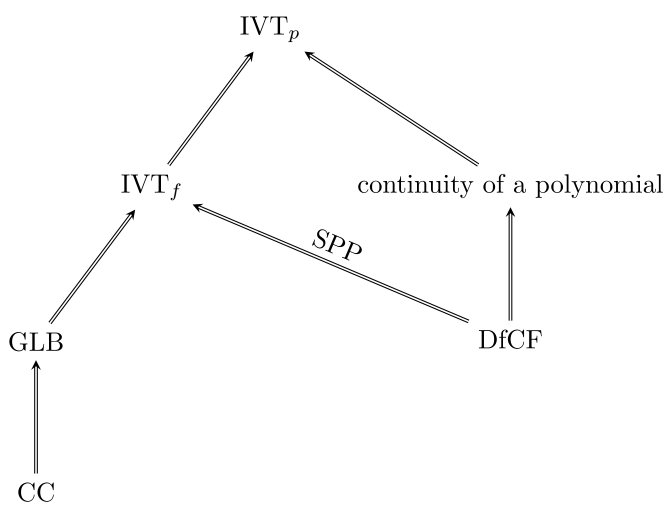
2. In modern mathematics, when continuity refers to the field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$, the greatest lower bound principle (GLB) characterizes the field up to isomorphism. When it refers to the ordered set $(\mathbb{R}, <)$, to get a categorical axioms system we have to add to GLB a form of the Archimedean axiom (AA).

Definitions of continuous function, $\varepsilon\delta$ and sequential, are formulated in the language for ordered fields, so by no means they depend on GLB. In [4], Heine proved their equivalence. Then, it was shown that equivalence depended on the Axiom of Choice. AA provides yet another insights into these definitions.

Still, the difference between continuous order and continuous function is not an absolute one as [2] proves that GLB is equivalent to the Intermediate Value Theorem for continuous function (IVT_f).

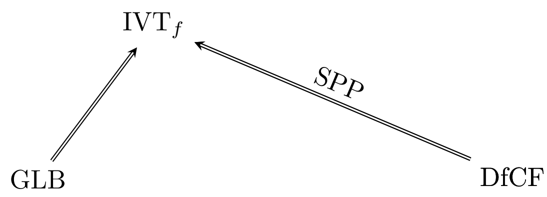
3. In [3], Bolzano seeks to prove IVT for polynomials (IVT_p). Although he also proves IVT_f, it plays but a role of lemma in the proof of IVT_p.

The diagram below shows the general plan of [3]; CC, SPP and DfCF stand for Cauchy completeness, sign preserving property, and definition of continuous function, respectively.



Bolzano attempts to prove CC and then derives GLB from it. Leaving aside possible circularity or mistakes, we argue that his most insightful contribution is the very formulation of GLB and DfCF.

4. Modern calculus adopted the following part of the Bolzano's proof.



Since IVT$_p$ easily follows from IVT$_f$, it is no longer considered to be a crucial proposition. However, IVT$_p$ does not depend on GLB as it holds in real closed fields.

5. In *Preface*, Bolzano criticizes "mechanical" and "geometrical" proofs of IVT$_p$. The first is flawed by its appeal to "the concept of the continuity of a function with the inclusion of the concepts of time and motion". Therefore he presents his DfCF. The second is circular, since it relies on the "general truth, as a result of which every continuous function" has IVT property.

We argue that there is the taken-for-granted assumption in Bolzano's argument that any line can be represented by some function. Marked by that duality of line and function, [3] belongs to the tradition (initiated by [4], developed in [5]) under which an analytic formula represents function, diagram represents line, and the relation between function and line is guaranteed by some non-mathematical conditions. That duality was covered by the arithmetisation of analysis, and then completed in the set-theoretic foundations of mathematics. Under the set-theoretic definition of function, a line is the graf of a function; when line is identified in $\mathbb{R}^n$ its continuity is related to GLB. On the other hand, the continuity of function is related to DfCF and echoes its "mechanical" provenance.

References

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