

Simplicity, Generality, and Economy: Philosophical Background of the *RAB*

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§1: Bolzano's Grand Project

From the early *BD*¹:

This much seems to me certain: in the realm of truth, i.e., the collection of all true judgements [read: propositions, SR], a certain *objective connection* prevails, which is independent of our accidental *subjective recognition* of it. As a consequence of this some of these judgements are the grounds of others and the latter are the consequences of the former. (*BD* II, 39-40)

By a *scientific proof* of a truth [we understand] the representation of the *objective dependence* of it on *other truths*, i.e., the derivation of it from those truths that must be considered as *the ground for it* [...] while the truth itself, in contrast, must be considered as their consequence. (*BD* II, 64)

Such *scientific proofs* are *explanatory*; they show not only *that*, but also *why* their theorem holds. The project of the *Purely Analytic Proof* (*RAB*²) is to provide precisely such a proof:

Thus all the proofs so far of the proposition which forms the title of this paper are inadequate. Now the one which I submit here for the judgement of scholars contains, I flatter myself, not a mere *confirmation*, but the *objective grounding* of the truth to be proved, i.e. it is strictly scientific. (*RAB*, 260)

NB: The inadequacy of the “proofs so far” does not stem from them being *logically* defective (at least not exclusively). Rather, they don't explain the theorem of interest.

§2: The Pure Theory of Grounding

Topic-neutral properties of grounding (\approx criteria for explanations in general)

- non-circularity: no truth grounds itself (*WL*,³ §§204, 209)
- factivity: only truths figure as grounds/consequences (*WL*, §205)
- non-monotonicity: grounding is not preserved under (*WL*, §207)
- uniqueness (?) (*WL*, §206)

¹*Beyträge zu einer begründeteren Darstellung der Mathematik*=Bolzano (1810a); I follow Steve's translation from Russ (2004).

²Bolzano (1817), pagenumbers refer to Steve's translation in Russ (2004).

³*WL*=Bolzano (1837).

§3: The Impure Theory of Grounding I: Closure and Simplicity

A good deal of what I have said in this section is generally accepted. For if it is said that a priori truths cannot be proved from experience, it can only mean that the objective ground of a purely conceptual truth cannot lie in intuitional propositions. And if it is said that in a truly scientific exposition we must proceed from the more general to the more specific, then this can derive only from the notion that the more general and simpler truths are the ground of the more particular and more complex ones. (*WL*, §221 [II.388], underlining SR)

Closure. Conceptual truths can only be grounded in conceptual truths. (*WL*, §221.1) This excludes explanatory proofs of theorems from pure mathematics that employ truths which include what Bolzano calls *intuitions* (i.e. ideas that directly refer to concrete particulars in space and/or time).

Simplicity. In *RAB*, Bolzano appeals to the idea that ground-revealing proofs proceed from the simple to the more complex. In the foreword, he dismisses a proof of the theorem in question terms of the Fundamental Theorem of Algebra (FTA): while the proof is logically flawless “the fact remains that such a derivation could not be called a genuinely scientific *demonstration* (*Begründung*), in that [FTA] clearly expresses a much more complex proposition than our present one” (*RAB*, 258).

But what does it mean, exactly, that the order of grounding proceeds from simpler to more complex truths? We find two explications of this idea in Bolzano’s mature work:

1. *Numerical*: If ϕ helps ground ψ , then the number of constituents into which ϕ can be decomposed is smaller than or equal to the number of constituents into which ψ can be decomposed. (*WL*, §221.2)
2. *Contensive*: If ϕ helps ground ψ , then ϕ contains no simple constituent that does not occur in ψ as well. (*WL*, §609; *ML*, §17; Bolzano (1813, 162))

The contensive criterion seems to be related to the ideal of *purity*: scientific proofs must not make use of notions *alien* to the theorem. (Bolzano appeals to that idea in *RAB*, 254.) Nonetheless Bolzano remained skeptical whether the contensive criterion holds generally; perhaps because this would entail that most *Modus Barbara* syllogisms are non-explanatory.

§4: The Impure Theory of Grounding II: Generality

In *RAB*, Bolzano *seems* to appeal to the idea that ground-revealing proofs have to proceed from more to less general propositions (in connection with the prohibition to use material *alien* to the theorem in an explanatory proof; cf. *RAB*, 254-5). Indeed, the claim that ground-revealing proofs move from more to less general truths might be thought to follow from to

the contentive simplicity-criterion on the assumption of the so-called *Canon of Reciprocity* (\approx content and extension of a concept vary inversely). However, we have seen that the contentive criterion might not hold always. Moreover, Bolzano rejects the *Canon of Reciprocity* in his mature writings (cf. *WL*, §120).

So how does the notion of generality figure in his account of scientific proofs? Bolzano *rejects* the following claim:

(GC) If ϕ helps grounds ψ , then ϕ is at least as general as ψ .

This is a good thing because (GC) would exclude explanatory proofs that have the following form:

$$\begin{array}{l} \text{Every } x \text{ that is } F \text{ is } G. \\ \text{Every } x \text{ that is } F' \text{ is } G. \\ \hline \text{Every } x \text{ that is } F \text{ or } F' \text{ is } G. \end{array}$$

But Bolzano takes such proofs generally to be explanatory (*WL*, §227). Moreover, (GC) would exclude proofs by induction from being explanatory. Most importantly, in a couple of passages, Bolzano directly rejects (GC) (cf. *WL*, §525 [IV.262] and §608).

Instead of (GC), Bolzano accepts the following principle:⁴

(GG) If the truths Γ ground ψ , then there is no set of truths Γ' such that (i) ψ is logically deducible from Γ' (ii) no $\phi \in \Gamma'$ is redundant for the deduction of ψ , (iii) $\psi \notin \Gamma'$, and (iv) Γ' is more general than Γ . (*WL*, §221.6)

This is compatible with the aforementioned kinds of proofs being explanatory, and it allows proofs by induction to be explanatory. A problem is that it remains unclear how to spell out the notion of comparative generality as applied to sets of truths.

§5 THE IMPURE THEORY OF GROUNDING III: ECONOMY

There is a set of criteria to which Bolzano does not explicitly appeal in *RAB*, but that is of the foremost importance for his conception of grounding for conceptual truths. Those criteria all centre around the idea that ground-revealing proofs exhibit a maximum of *economy*. They make crucial use of Bolzano's explication of the notion of deductive validity. The general idea is as follows:

- Among all deductively valid arguments for a given truth ψ , the explanatory one is the most economical one.

'Economical' here is explained in terms of possessing the smallest set of premises, each of which is the simplest one among all logically equivalent ones. In particular, Bolzano proposes two principles:⁵

⁴This is a simplified version, for more details see Roski (2017, ch. 4.4).

⁵Again, I am simplifying a bit. For more details, see Roski and Rumberg (2016).

(E1) If Γ grounds ψ , then there is no alternative set of premises Γ' such that (i) ψ is logically deducible from Γ' , (ii) no $\phi \in \Gamma'$ is more complex than ψ , and (iii) Γ' contains fewer propositions than Γ . (*WL*, §221.5)

(E2) If ϕ helps grounds ψ , then there is no other truth ϕ' that is logically equivalent to ϕ and less complex than ϕ . (*WL*, §221.4)

This yields a general heuristic for finding explanatory proofs. Consider all deductively valid arguments for ϕ . Then, first, get rid of all arguments that make use of more premises than strictly necessary. Second, among all remaining arguments, get rid of all those that make use of unnecessary conceptual material. (This should give us a means to check whether Bolzano's purely analytic proof is explanatory by his own standards. Is there an alternative proof that can reasonably said to be more economical?) In line with the Contensive Criterion, moreover, one should aim at not using concepts alien to the theorem in question.

An analysis of the given proposition that extends to its simple parts, insofar as we are able to perform it, must be our first business in this problem. Next, we must construct from the parts we have discovered in the proposition M propositions that are simpler or at least not more complex than M , and also constituted so that M is deducible from them. If in doing this we find it necessary to have recourse to concepts not contained in M , we must seek to reduce their number as far as possible. Only when we have succeeded in this way in assuring ourselves that the truths A, B, C, D, \dots from which M is deducible, are each simpler or at least no more complex than M , and taken together constitute a simpler collection than any other from which M may be deduced may we permit ourselves to look upon the former as the ground of the latter. (*WL*, §378)

Bolzano also contemplated the idea that (E1) and (E2) might not only be necessary conditions for a set of truths being the ground of a given conceptual truth, but that they also might form the essential part of a sufficient condition for grounding to hold among conceptual truths.

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