Bolzano's Infinite Quantities

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Abstract

In his *Foundations of a General Theory of Manifolds* from 1883, Georg Cantor praised Bernard Bolzano as a clear defender of actual infinity who had the courage to work with infinite numbers. At the same time, he sharply criticized the way Bolzano dealt with them; he had never developed the general concept of cardinality. While Cantor's concept was based on the existence of a "one-to-one correspondence", Bolzano insisted on Euclid's Axiom that "the whole is greater than a part". Cantor's set theory has eventually prevailed, and became a formal basis of contemporary mathematics, while Bolzano's approach is generally considered a step in the wrong direction even by Bolzano scholars.

In this paper, we demonstrate that the fragment of Bolzano's theory described in his *Paradoxes of the Infinite* from 1848 retaining the part-whole principle can be extended to a consistent theory in several distinct ways. Bolzano's infinite series are interpreted as infinite sequences of partial sums. There is a slight conceptual distinction between treating elements of R^N as Bolzano's quantities representing single, exactly given quantities, and treating them as classical sequences. The two viewpoints are formally equivalent, but can lead to a somewhat different way of thinking about such objects.

In conformity with Bolzano's condition of associativity and commutativity we define equality of sequences so that the corresponding terms of sequences are equal from a sufficiently large index. The order is introduced on the same principle. In fact, we use the Fréchet filter here. Addition and multiplication of sequences is defined pointwise. We obtain a partially ordered ring of infinitely small, finite and infinitely great quantities R^0. All Bolzano's propositions concerning infinite quantities stated in his *Paradoxes of the Infinite* are valid in this interpretation. Moreover, the structure R^0 can be used as a basis for a simplified version of infinitesimal calculus as was recently demonstrated by Terence Tao in his paper *A cheap version of non-standard analysis* (Tao 2012).

The structure R^0 contains alternating sequences. Consequently, it is neither linearly ordered nor it is a field. Bolzano's further condition of the existence of a *formation rule* can be interpreted so that the terms of sequences must be expressed as polynomials. The structure of *polynomial sequences* is a linearly ordered integral domain representing Bolzano's finite and infinitely great quantities. Nevertheless, no polynomial sequences represent infinitely small quantities, therefore this structure is unsuitable for mathematical analysis.

The only way to obtain a substructure of R^0 being a linearly ordered field containing infinitely small and infinitely great quantities is via an ultrafilter. It is the same construction, but instead of the Fréchet filter we use a non-principal ultrafilter. The ultraproduct R* is a consistent mathematical framework for non-standard analysis. (Robinson 1966). Ultrafilters and their wonderful properties are the result of modern logic of the 20th century. We need the axiom of choice to prove their existence. Bolzano, of course, could not have known that.