# The Germanic Development of the Pre-Modern Notion of Number From c. 1750 to Bolzano's *Rein analytischer Beweis*

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## 1. The genesis of the thesis

My initial research was about the development of Cantorian set theory. Supervised by professor José Ferreirós, the general question of my project was how Cantor's new methods and concepts emerged from traditional ones. So, despite the fact that, as in the case of all mathematicians, his extra-mathematical motivations were something to consider, I was mostly interested in his mathematics. That way, I started by studying those works prior to his series on "infinite, linear point-manifolds", among them his 1874 work "On a Property of the Collection of All Real Algebraic Numbers", in which he proved a) the denumerability of algebraics and b) the non-denumerability of –our– $\mathbb{R}$.

A key aspect of the aforementioned paper is that it contains a proof of the Bolzano-Weierstrass principle (i.e. an infinite sequence of nested closed intervals of real numbers must have a nonempty intersection) on the basis of the theorem according to which "if a magnitude $x$ grows constantly but not beyond all limits, then it approaches a limit value" [Dedekind, 1872, 29-30]. In other words, Cantor opted to prove that principle according to "Dedekind's preferred proposition that every monotonically increasing, bounded sequence of real numbers has a limit" [Ferreirós, 1996, 181-182], instead of applying it directly. A procedure that, it must be said, was not the same as in his –unpublished– proof sent to Dedekind in his correspondence of December 1873 and that in fact was strange in him, given the importance attributed by Weierstrass and his students to that principle.

On the one hand, that led me to pay more attention to the particularities of the theories of Cantor and Dedekind and their definitions of real numbers. On the other hand, that led me to study mathematicians who were even prior to Weierstrass, in order to better understand the mathematics taught to Cantor. We decided to focus on French and Germanic mathematics, and in particular on three authors, namely, Lagrange, Cauchy (aided by professor Marco Panza) and Bolzano (aided by professor Hourya Benis Sinaceur).

The idea was to understand the proposals of those mathematicians and certain results of each of them that were crucial for later mathematics, to later study other authors of the first half of the 19$^{th}$ century linked directly to Cantor. However, two details in the work of Bolzano caught my attention. Firstly, the fact that in his posthumously published *Paradoxes of the Infinite* he referred to the "set of all numbers (the so-called natural or whole [...])" [Bolzano, 1851, 20], which meant that in the mid-19$^{th}$ century that concept was common. Remembering a question that F. William Lawvere asked professor Ferreirós at the CLMPS about the origin of the natural numbers, I thought it was worth to trace the emergence of that concept, something that professor Hourya encouraged me to do.

Secondly, when reading his "Purely Analytic Proof" it seemed to me that even though Bolzano's mathematics appeared to be similar to later mathematics, there were some details that deserved careful attention. That is to say, except those details, Bolzano's early mathematical proposal seemed to be compatible with an early stage of modern real analysis: certainly, strictly speaking, what is found in that work is not the Bolzano-Weierstrass theorem, just as his terminology is not the modern one (as neither was the one of, for example, Méray, Weierstrass, Cantor), but his words, concepts, procedures and results can be interpreted as going in the modern direction. And, yet, the question was what Bolzano was doing or what he was trying to do.

As a consequence, we decided that it was necessary to try to understand Bolzano's 1817 work, those 'curiosities' included. This required studying his works published until 1817, as well as to study the mathematical context in which he grew up. A context, that of Germanic mathematics, usually reviled compared to French and interpreted "in the shadow of Wolff". A context in which, given the importance of the universities of Göttingen and Halle, the authors that I chose to study were Segner, Kästner and Karsten, the math teachers at those universities; teachers who, at least to some extent, account for the conceptual and practical changes between Wolff and the Germanic mathematicians of the late $18^{th}$ century and early $19^{th}$ century.

That was how the thread of my research went from being the set-theoretical concepts and methods, to be the notion of numbers, as well as from being focused on Cantor to being focused on Bolzano. However, the fact that the thesis is about the development of "the pre-modern" and not "the modern" notion of number has to do with the results of such research, as described below.

## 2. The structure of the thesis

## 3. The thesis on Bolzano's early mathematical works

The objective of my thesis is to discern Bolzano's notion of number in his early mathematical works (1804-1817). As a starting point, consider what Bolzano says in the following three fragments:

I. The best procedure might well be to count as higher mathesis only that in which the concept of an infinity (whether infinitely large or small), or of a differential, appears. At the present time this concept has not yet been sufficiently explained. If, in the future, it should be decided that the infinite or the differential are nothing but symbolic expressions just like $\sqrt{-1}$ and similar expressions, and if it also turns out that the method of proving truths using purely symbolic inventions is a method of proof which is indeed quite special, but is always correct and logically admissible, then I believe it would be most appropriate to continue to refer the concept of infinity, and any other equally symbolic concept, to the domain of higher mathematics. Elementary mathesis would then be that which accepts only real concepts or expressions in its exposition— higher mathesis that which also accepts purely symbolic ones. [1810, 30-31]

II. Likewise, instead of the so-called infinitely small quantities I have always made use, with equal success, of the concept of those quantities, which can become smaller than any given quantity, or (as I sometimes call them to avoid monotony but less precisely) quantities, which can become as small as desired. I hope no one will mistake the difference between quantities of this kind, and what is otherwise thought of under the name 'infinitely small'. The requirement of imagining a quantity (I am thinking of a variable quantity) which can always become smaller than it has already been taken, and generally can become smaller than any given quantity, really contains nothing that anyone could find objectionable. On the contrary, surely anyone must be able to see that there are very often such quantities, in space as well as in time? On the other hand, the idea of a quantity which cannot only be assumed to be smaller, but is really to be smaller than every quantity, not merely every given quantity but even every alleged, i.e. conceivable, quantity, is this not contradictory? [1816, V]

III. According to the known rules of multiplication, the second, third, fourth ... power of every quantity consisting of two or more parts, like $a + b$, or $a + b + c + d + \cdots$, can be expanded in a series whose individual terms contain nothing but powers of the individual parts $a, b, c, d, \ldots$ or products of such powers, possibly also multiplied by a specific number. [...] This observation gives rise to the idea of whether perhaps in general every function consisting of two or more parts of the form $(a + b)^n$, $(a + b + c + d + \cdots)^n$, where $n$ may denote any kind of whole number, indeed also a fractional, irrational or negative quantity, can be expanded in a series which, like the above, contains nothing but powers of the individual parts $a, b, c, d, \ldots$ or products of such powers, possibly also multiplied by a quantity dependent merely on $n$. [1816, 1-2]

Fragment I corresponds to §17 of BD and has to do with a classification of general mathematics proposed by Michelsen [Michelsen, 1790 II, 135-134]. That Bolzano still conceives our imaginary numbers as "symbolic expressions" (*symbolischer Ausdruck*) at the time of RAB is clear from §75 (the last one) of BL, where he lists "imaginary expressions" (*imaginärer Ausdrücke*) among the concepts whose clear development is pending. This conception was not entirely strange at the time and can be found, for example, in [Cauchy, 1821: iij-iv & 173ff.] and [Jandera, 1830, XXIX]. It should be noted that, for Bolzano, while imaginary are "symbolic expressions", infinitely small and large quantities are not even that yet.

Fragment II corresponds to BL Preface and is where Bolzano defines his quantities $\omega$ as opposed to infinitely small quantities inasmuch as the latter is contradictory and the former is not objectionable. Bolzano uses this concept explicitly in [1817A] and implicitly in [1817B]. At the time, as Russ points out [Russ, 2004, 143-144], rejection of infinitesimals was not strange, as [Lagrange, 1797] and [Dubourguet, 1810] show. In addition, in the Germanic context of the second half of the $18^{th}$ century, the rejection of infinitely small quantities was very common [cf.

Chapter B]. On the other hand, alternatives similar to Bolzano's notion are found in, for example, [Carnot, 1797] and [Cauchy, 1821]. It is interesting to note his remark on the existence of variable quantities $\omega$ in space and time.

Fragment III corresponds to §1 of BL and it is a distinction between numbers and quantities that is found throughout the whole paper. This was common by the time and even Kant, in a reply to a letter from August Rehberg in the Autumn of 1790 [Kant, AA XI], defended that quantity $\sqrt{2}$ was not a number (it could be instantiated geometrically but not numerically), although it could be approximated by a sequence of fractions [cf. van Atten, 2012]. And, in his mathematical dictionary, Klügel defined *Irrational* as a "proportional concept" for quantities (*Verhältnissbegriff für Grössen*) [Klügel, 1805, 949]. Furthermore:

a) Ohm and, in a different way, Kronecker, also considered positive whole numbers as the only numbers [Ohm, 1822 I, XI; Kronecker, 1887, 338-339];

b) Hermann Hankel (a former student of Weierstrass) insisted that "Die Geometrie beweist die Existenz des Irrationalen" [Hankel, 1867, 59];

c) Charles Méray (who published an arithmetic theory of irrational numbers in 1869) declared that he reserved the term 'number' –"or quantity– for integers and fractions [Méray, 1869, 284];

d) Rudolf Lipschitz wrote to Dedekind that proportions "were enough to develop [...] the basic operations of calculus, which basically is the same thing that you carry out with your principle" [Lipschitz to Dedekind, 6.7.1876].

As for the negatives, as shown in Chapter B, the Germanic mathematicians of the second half of the 18th century and the beginning of the 19th century usually considered them as quantities numerically negatively expressed. Bolzano, in fact, constantly refers to them throughout BL as quantities with negative whole numerical value (*negativen ganz zähligen Grösse*).

Traditional narrative goes more or less like this: Bolzano's "Purely Analytic Proof" must be considered the starting for the modern theory of the continuum [cf. Ewald, 1999, 225-226], despite the fact that his work was incomplete, precisely because he lacked a definition of the continuum [cf. Stillwell, 2010, 22]. In other words, he must be recognized as one of the fathers of the arithmetization of analysis [cf. Klein, 1926, 56]. In favour of this interpretation one definition (of what nowadays we would call the continuity of a one real variable function) and three theorems and their respective results (his, we would say, criterion for the pointwise convergence of an infinite sequence; his, it is said, formulation of the Bolzano-Weierstrass theorem; and his proof of the theorem that entitles his work via the intermediate value theorem) are offered as evidence. That way, Hans Freudenthal wrote about the first pillar: "Bolzano's and Cauchy's definitions [of the continuity of a function] are equivalent. Bolzano's is far better; it is modern (though instead of $\delta$ and $\varepsilon$ he uses $\omega$and $\Omega$)" [Freudenthal, 1971, 380]. While Carl B. Boyer wrote about the third pillar: "proved by Bolzano [Bolzano-Weierstrass theorem], [...] it was the work of Weierstrass that made it familiar to mathematicians" [Boyer, 1968, 605]. In the middle, the second pillar is identified with the later criterion of Cauchy and praised but considered defective [cf. Bourbaki, 1971/2007, TG IV.72; Steele, 1950, 29-30].

Undoubtedly, the traditional narrative has in its favour not only the terminological and procedural similarity between Bolzano's mathematics and modern mathematics, but also accredited voices that go back to Weierstrass himself. To better understand such an interpretation, it is convenient to move the attention from the particular results of Bolzano to the underlying to arithmetization and modern real analysis. That is, before analysing if Bolzano proved Bolzano-Weierstrass theorem in §12 or if his $\omega$ were Weierstrassian $\varepsilon$ or at least Cauchy's $\alpha$ (interpreted as an anticipation of the epsilontic notion of limit), it is worth considering the foundations on which the characterization of real domain as totally ordered, dense, continuous rests:

- Conception of natural numbers as a domain;
- realization that a clear conceptual characterization of the continuity of the real-number domain was needed;
- establishment of fundamental concepts related to limits and derivation of the main theorems concerning those concepts;
- abandonment of the core notion of variable quantity, eventually replaced by syntactic variables and functions of a real variable;
- embracement of a purely mathematical domain of objects (numbers) antecedently given, static.

Now, the question is the following: Is the early mathematical proposal of Bolzano, if not one pointing towards the direction of modern real analysis, at least one that is non-contradictory (compatible) with it? The answer to this question urges the study of both what Bolzano says and what he does. After all, it is possible that some of his words and concepts were not entirely adequate from the modern perspective, but his way of thinking and doing mathematics is what ultimately enables such a link. Thus, for example, it is undeniable that his foundational and methodological concerns are moving towards the ground on which Weierstrass and others developed their proposals. This, although it was not usual at the time, was something that in some way or another Bolzano shared with some of his contemporaries [cf. Schultz, 1790; Michelsen, 1790; Klügel, 1798; Thibaut, 1805 & 1809; Gauss; Mayer, 1818; Ohm, 1822]. So, perhaps he was not completely alone in that enterprise, and maybe this was even motivated by –or originated because of– the Germanic reluctance to accept without further ado the 'foreign' or 'French' developments, but it is a fact that he embraced it as few –or, strictly speaking nobody else– did at the time and, not a minor detail, he did it in Prague at the beginning of the 19th century.

Nevertheless, the question remains and it is not only that he works with –positive– integers as the only type of numbers, with negative numbers as quantities with negative whole numerical value, with fractions as quantities susceptibles of numerical value, with irrationals (only once, and it seems that 'by mistake' or as a license, he referred to *irrationalen Zahl* [1816, 137]) as indeterminate quantities for which fractions could be used as means to determine them [1816; 1817A] and with imaginary as symbolic expressions. It is also that, as mentioned before, he explained in §75 of BL that he avoided certain cases involving the still unclear concepts of imaginary expressions, irrationals and mathematical opposites and: a) he works with "variable quantities" as core analytical objects; b) with $\omega$ as non-infinitesimal quantities (non-contradictory concept) [1816, 1817]; c) with determined segments of infinite series [1816, 1817A; cf. Russ, 2004, 144]; and, within the geometric-analytical framework set in [1817B], d) with "multitudes of infinite elements" [cf. 1804].

That said, the theses of my work are:

I. Major thesis: Bolzano's mathematical project until 1817 is pre- and not proto-Weierstrassian since, despite their apparent similarities, not only it is not an effort to create a "theory of $\mathbb{R}$ [...] using set-theoretic constructions [and] starting from the natural [or rational] numbers" (... arithmetization), but above all it still features traits of mathematical notions and practices that are heavily deviant from the Weierstrassian and later ones of modern real analysis.

II. Minor thesis: Bolzano's variable quantities $\omega$ are neither numbers (Weierstrassian $\varepsilon$) nor quantities that tend to zero in a strictly dynamic way (Cauchy $\alpha$), but quantities that can become smaller than any given one and, therefore, might represent an attempt to work in 'finite' terms.

To make it even clearer, from my point of view there are some aspects of Bolzano's early mathematical proposal that should not be neglected, among them, first of all, his careful definition of quantities $\omega$ (with $\Omega$ as an "algebraic sum or difference of a finite multitude of quantities $\omega$") [1816; 1817A]. Because, if one assumes that Bolzano is congruent with his mathematics, quantities $\omega$ cannot be:

a) Weierstrassian $\varepsilon$ and, ultimately, numbers [cf. Weierstrass 1886/1988, 57ff.], since that would contravene not only his notion of 'numbers' but also his not yet syntactic notion of 'variable quantities';
b) Cauchy's $\alpha$ [Cauchy, 1821, 19], understood as an anticipation of Weierstrassian $\varepsilon$ (traditional interpretation) or as implicitly kinetic quantities [cf. Sinaceur, 1973];
c) "continuously decreasing" (*continuellement décroissante*) quantities [Carnot, 1797, 19], explicitly kinetic;
d) infinitely small quantities.

More over, if one assumes his congruence or, at least, his effort to be congruent, his 1816 remark about the fact that "there are very often such quantities, in space as well as in time" [1816, V], once again does not fit with the syntactic conception of variables within the later analytical framework and indeed implies assuming space and time as continuous, something that later mathematicians would consider an axiom [cf. Cantor, 1872; Dedekind 1872]. And, last but not least, his denomination of irrationals as quantities, not numbers, and his insistence on their not entirely determined value for which, nevertheless, fractions could be used [1816, 1817A], is contrary to the crucial embracement of infinity required for the development of the modern continuum.

At the same time, it should not be neglected either that Bolzano attempts to work in a purely analytic manner, which implies working in non-geometric terms or without geometric considerations [cf. 1810], and argues that general mathematics deals, properly speaking, with "objects of thought" (*Gedankendinge*) [1810, 6; cf. 1804, 48]. In addition to which, his conception of mathematics as the science of "general laws (forms) to which things must conform in their existence" [1810, 11], even if by 1814 had changed to "the science of abstract quantities" [cf. Blok, 2017, 172], things that existed not only "independent of our consciousness" but also "in our imagination", moves along a path that could end in the project of a purely mathematical (or purely logical) foundation of the concept of a continuous domain.

The point is, however, that even though, as denounced by Richard Dedekind in 1872 (*veränderlichen Grösse*) and 1888 (*messbaren Grössen*), remnants of ideas alien to modern analysis can be found in certain expressions in [Méray, 1869], as well as in the works of Weierstrass and Cantor, who still used the appellative "numerical quantities" (*Zahlengrössen*) to refer to rational and irrational numbers within a pretty modern abstract conception [cf. Cantor, 1872/1932, 97; Weierstrass, 1878/1988, 7, 8 & 40), in Bolzano it is not a case of mere terminology. His distinction between quantities and numbers in his early works, allow me to insist, is not merely verbal, but conceptual and practical and it is rooted in prevailing notions and practices among Germanic mathematicians. But once he got rid of that 'ballast', the initial tension around the infinite disappeared and he was able to work with numbers within a pretty modern abstract conception [RZ; cf. Russ & Trlifajová, 2016]. A standpoint which I think that can be compatible with [Loeb & Roski, 2014], [Russ & Trlifajová, 2016] and even [Trlifajová, 11/7/2017], insofar as my guess is that Bolzano's early mathematical proposal would not necessarily have led to the Archimedean continuum (but that is another story).

So, what are quantities $\omega$? My interpretation is that they are either Cauchy $\alpha$ in the sense of implicitly kinetic, that is to say, quantities that, after all and against his will, are implicitly kinetic and appeal to infinite; or, they are something else that seems strange to modern understanding because it demands thinking about the continuum in a way that is perhaps 'counterintuitive' or even 'paradoxical', untenable, to us, modern as we are. As a consequence, while I consider that the attempt of Rusnock and Kerr-Lawson somehow accounts for the first scenario [Rusnock & Kerr-Lawson, 2005, 306], but without addressing its implicitly kinetic character, I think it would be worthwhile to work in an entirely faithful way to the definition of quantities $\omega$ given by Bolzano. Which, in any case, would still require an embrace of infinity against which Bolzano was by 1817, in order to sustain the whole traditional interpretation of his "Purely Analytic Proof".

4. Pending research tasks

- Bolzano's manuscripts and Archives material.
- Change between his early works and [RZ] on infinity. [cf. Russ & Trlifajová]
- His notion of Formen [1810, 1816] in relation to mathematicians linked to the Combinatorial School, for whom combinatorics was at the core of maths, "science of the forms [of quantities]" was a common definition and the manipulation of the infinite in 'finite' terms was commendable. The notion of Function in Bolzano's work and in relation to Formen.
- His late works in the context of Germanic mathematics & arithmetization, as well as pre-set and pre-topological ideas in his early works.
- Better understanding of changes in the way of doing and thinking about mathematics in the late 18th and early 19th century, focused on Bolzano's Germanic teachers and contemporaries.