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ABSTRACT This paper is devoted to Bolzano's theory of grounding (*Abfolge*) in his *Wissenschaftslehre*. Bolzanian grounding is an explanatory consequence relation that is frequently considered an ancestor of the notion of metaphysical grounding. The paper focuses on two principles that concern grounding in the realm of conceptual sciences and relate to traditionally widespread ideas on explanations: the principles, namely, that grounding orders conceptual truths from simple to more complex ones (Simplicity), and that it comes along with a certain theoretical economy among them (Economy). Being spelled out on the basis of Bolzano's notion of deducibility (*Ableitbarkeit*), these principles are revealing for the question to what extent grounding can be considered a formal relation.

KEYWORDS Bolzano, (formal) grounding, deducibility, simplicity, economy, explanation

INTRODUCTION

NOWADAYS, BOLZANO IS PROBABLY MOST KNOWN for his definition of a consequence relation that bears close resemblance to the Tarskian notion of logical consequence and anticipates the latter by about a hundred years. Bolzano calls this relation *deducibility* (*Ableitbarkeit*). But deducibility is not the only consequence relation we find in Bolzano's writings—it is not the only sense in which Bolzano claims that certain propositions *follow from* or are *consequences of* other propositions. Of much more importance for his general philosophical project in his main work, *Wissenschaftslehre*,[1] is another consequence relation that Bolzano calls *grounding* (*Abfolge*)—a relation that obtains between *grounds* (*Gründe*) and their *consequences* (*Folgen*).

Unlike deducibility, grounding is only defined among true propositions and has certain explanatory properties. Grounds, Bolzano argues, are specified in

[1]We will refer to passages of the *WL* in the following format: <paragraph> [<volume>.<page>]. Our translations follow Rusnock and George, *Theory of Science* (the first complete English translation of the *WL*).

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answers to explanation-seeking why-questions; that is, in answers to questions as to why a given proposition is true, as opposed to questions concerning evidence for the proposition's truth.[2] Relatedly, Bolzano takes it that grounding underlies the classical distinction between explanatory and non-explanatory proofs, that is, between proofs that show *why* a proposition is true and proofs that merely show *that* a proposition is true, which goes back to Aristotle's *Posterior Analytics*.[3]

Even though Bolzano carefully distinguishes grounding from deducibility, both notions are not unrelated. In particular, Bolzano tries to capture a number of central properties of the explanatory consequence relation of grounding on the basis of the formal consequence relation of deducibility. If we call an argument in which the conclusion is deducible from the premises *deductively valid*[4] and an argument in which the premises form the ground of the conclusion *explanatory*, we may put Bolzano's main question in this connection as follows: under what conditions is a deductively valid argument explanatory? When it comes to grounding in the a priori or conceptual sciences, two ideas of Bolzano's theory of grounding play a pivotal role in answering this question. These ideas can roughly be summarized as follows:

(Simplicity) Grounding orders truths from simpler to more complex ones.

(Economy) Grounding orders truths in the most economical way.

In fact, Bolzano even goes so far as to suggest that, when focusing on a priori or conceptual sciences, the notion of grounding might be captured entirely in terms of deducibility constrained by a number of claims that correspond to (Simplicity) and (Economy).

Although the importance of these claims for Bolzano's views on grounding has frequently been emphasized,[5] they have so far not been investigated in much detail. The prevailing literature on Bolzano's theory of grounding has focused primarily on the basic relational properties of grounding[6] and its relation to modern proof-theoretic ideas.[7] These aspects of the theory allow us to draw a

[2]See *WL*, §177. On this topic, cf. Tatzel, "Ground and Consequence," and Schnieder, "Causation." Of course, an answer to an explanation-seeking why-question concerning a given proposition may just as well indicate evidence for the proposition's truth.

[3]Cf. *Post. An.*, 71b20–22, 78a; *WL*, §198 [II.341]; §525 [IV.261]; and Bolzano, "Lehrart," §14. On this topic, consider Berg, *Bolzano's Logic*, 161–63; Betti, "Explanation in Metaphysics," 289–90; Buhl, *Ableitbarkeit und Abfolge*; Centrone, "Begründungen," §2; de Jong, "Analyticity," 342–45; Mancosu, "Bolzano and Cournot," 439–40; Roski, *Bolzano's Notion of Grounding*, ch. 6.5; Rumberg, "Normal Proofs," 425; Rusnock, *Bolzano's Philosophy*, 22–25.

[4]As we shall see below, this notion of deductive validity is somewhat broader than the contemporary notion that goes by the same name.

[5]In an anonymously published summary of the *WL*, Bolzano himself stresses the importance of these principles for his theory; cf. Bolzano, Übersicht, 69–70. As is clear from Centrone, "Strenge Beweise," §§6–7, related principles already figure importantly in Bolzano's early work on grounding. Their relevance in the broader context of Bolzano's methodology has also been stressed by Betti, "Explanation in Metaphysics," and earlier by Buhl, *Ableitbarkeit und Abfolge*. Betti argues that the principles witness Bolzano's indebtedness to a traditional ideal of scientific rationality, which Betti and de Jong ("Classical Model of Science") dub the *Classical Model of Science*. The present paper complements that of Betti in that it investigates the theoretical details that underlie Bolzano's broader methodological views she focuses on.

[6]See Berg, *Bolzano's Logic*, and Tatzel, "Ground and Consequence."

[7]See Rumberg, "Normal Proofs"; Tatzel, "Proving and Grounding"; Centrone, "Begründungen," 22; and Centrone, *Early Husserl*, 127.

close parallel between Bolzano's ideas and the currently hotly debated notion of metaphysical grounding. In fact, Bolzanian grounding is frequently considered a predecessor of the modern metaphysical notion.[8] Moreover, Schnieder has recently provided a detailed discussion of Bolzano's analysis of causality in terms of grounding and thus, as it were, of the part of his theory that is mainly relevant for explanation in the empirical sciences.[9] The present paper instead deals with the part of Bolzano's theory that is relevant for non-causal explanations in a priori or conceptual sciences, a topic that is at the core of Bolzano's interest in the notion of grounding.[10]

What makes a close investigation of Bolzano's claims concerning (Simplicity) and (Economy) particularly interesting is the fact that those claims can be related to certain intuitions concerning explanations that have been widespread throughout history and are still influential—the intuitions, namely, that complex truths are to be explained by simpler ones, and that the explanatory order goes in hand with some kind of theoretical economy. A thorough discussion of Bolzano's respective claims thus helps us to acquire a much more informative picture of the particular conception of explanation that can be ascribed to him.[11] Since many of the claims we shall investigate moreover specify properties of grounding in terms of the formal consequence relation of deducibility, our investigation also shows to what extent grounding can be considered a formal relation and whether it can be partially captured in terms of formal rules. While the latter is a common assumption of current approaches to grounding,[12] we will see that it is not entirely unproblematic against the background of Bolzano's theory.

The paper is structured as follows: we first provide the nuts and bolts of Bolzano's conceptual framework that are necessary for understanding his theory of grounding and go over the definition of his notion of deducibility (section 1). Afterward, we introduce the basic relational properties of grounding (section 2). In section 3, we turn to the claims of Bolzano's theory that correspond to (Simplicity) (section 3.1) and (Economy) (section 3.2). In this context, we discuss a number of exegetical as well as systematic problems surrounding those claims and offer at least partial solutions. The final section (section 4) is concerned with Bolzano's attempt to provide a sufficient condition for grounding on the basis of claims that relate to (Simplicity) and (Economy) and with the question of the extent to which grounding can be considered a formal relation.

[8]Correia and Schnieder, "Opinionated Introduction," provides an overview of the current debate on metaphysical grounding and emphasizes Bolzano's pioneering role.

[9]See Schnieder, "Causation." Here and below, when we speak about explanations, we mean an *objective* notion of explanation in the sense described e.g. by Ruben, *Explaining Explanation*, 7–8.

[10]As has often been pointed out, Bolzano's groundbreaking results in mathematical analysis, for instance, were motivated by his views on grounding in the a priori sciences; cf. Mancosu, "Bolzano and Cournot"; Rusnock, *Bolzano's Philosophy*; Behboud, *Bolzanos Beiträge*. Moreover, as Centrone has pointed out, Bolzano's discussion of these ideas also influenced early Husserl's view on a priori sciences; cf. Centrone, *Early Husserl*, 127; Centrone, "Begründungen," 21–22.

[11]A modern approach to explanation that seems to be very much in the same spirit is the unificatory theory of explanation; cf. Friedman, "Scientific Understanding"; Kitcher, "Explanatory Unification." Further historical references will be provided below.

[12]See e.g. Correia, "Logical Grounds."

1. BASIC CONCEPTS OF BOLZANO'S LOGIC

A good starting point to introduce the basic notions of Bolzano's logic is the notion of *proposition* (*Satz an sich*), which Bolzano considers a fundamental, undefined notion. Propositions are abstract objects that are either true or false and figure as the content of judgments and beliefs and as sentence-meanings. In many respects, they are comparable to Fregean thoughts (*Gedanken*).[13] Unlike Frege, however, Bolzano adheres to a traditional conception concerning the *form* of propositions. He assumes that every proposition is of the form '*A* has (the property) *b*' and that each sentence that expresses a proposition can be paraphrased accordingly.[14] For the topic of this paper, we need not concern ourselves with this particular assumption. When discussing concrete examples, we will mostly follow the linguistic surface structure—as Bolzano does himself most of the time.

Bolzanian propositions are complex, structured entities that are ultimately composed of sub-propositional parts, which he calls *ideas in themselves* (*Vorstellungen an sich*; we henceforth omit 'in themselves').[15] Ideas are in turn either simple or built up from simple ideas. Some ideas have one or several objects or properties *falling under* them, whereas other ideas have nothing falling under them.

A distinction among true propositions that will become important in what follows is the distinction between *conceptual truths* (*Begriffswahrheiten*) and *intuitional truths* (*Anschauungswahrheiten*) as we restrict our investigations below for the most part to grounding among conceptual truths. We cannot go into the intricacies of how precisely Bolzano draws this distinction in this paper.[16] In order to get an idea of what difference Bolzano tries to capture, suffice it to say that conceptual truths are typically truths that do not concern any particular individual in space and time, whereas intuitional truths always concern such individuals. If a conceptual truth *does* concern individuals in space and time, it does so only generically (by quantifying over them, or by mentioning collections that contain them); and if a conceptual truth concerns any *specific* object, the object will be an abstract one (a number, a proposition, and so forth). Typical examples for conceptual truths in Bolzano's sense are accordingly truths of pure mathematics, logic, metaphysics, and similar non-empirical disciplines—truths that can be expressed without using proper names or indexicals. Typical examples for intuitional truths, on the other hand, are the truths such as [I am here now] or [This rose smells sweet]—truths that generally need to be expressed by using proper names or indexicals.[17]

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[13]For an illuminating comparison, cf. Künne, "Propositions."

[14]See *WL*, §127. Betti, "Bolzano's Universe," contains a recent discussion of this assumption.

[15]On the translation of '*Vorstellung*' as 'idea,' see Rusnock and George, *Theory of Science*, I. xxvi–xxvii.

[16]See *WL*, §133. Textor, *Propositionalismus*, provides the most comprehensive study of the distinction. Textor, "Conceptual and Intuitive Truth," contains a recent and more accessible discussion.

[17]As usual in Bolzano scholarship, we denote propositions and ideas by enclosing the linguistic items that express them in square brackets. Note that even though some conceptual truths are analytic, Bolzano does not assume that all of them are. The truth [This rose is either red or not red] is analytic according to Bolzano but not a conceptual truth; cf. *WL*, §197 and Textor, "Logically Analytic." Note further that Bolzano does not want to commit himself to the view that *all* conceptual truths are knowable a priori; cf. *WL*, §133 [II.36]. This may be because he suspects that some truths, including conceptual ones, may be generally unknowable for human beings and thus in particular not knowable a priori; cf. Lapointe, "A Priori Knowledge," 275.

A technique Bolzano uses to define numerous properties of propositions and ideas is his so-called *method of variation*. The idea underlying this method is rather straightforward. Variation closely corresponds to substitution. Properties of, and relations among, propositions and ideas are defined in terms of their behavior under simultaneous uniform substitution of some of their constituents.[18] To consider a *variant* of a given proposition φ with respect to one of its constituent ideas v simply means to consider another proposition $\varphi(w/v)$ that differs from φ in that it has the idea w in place of v.[19] This idea can naturally be extended to the simultaneous substitution of more than one constituent idea in more than one proposition. If the variation of certain constituent ideas in a proposition results in a true proposition, we say that the substitution makes the proposition true, and similarly for collections of several propositions.[20]

Bolzano's notion of *deducibility*, which we mentioned in the introduction, is defined in terms of variation:[21]

> *Deducibility*: A proposition ψ is deducible from a collection of propositions Γ with respect to a collection of ideas $v_1,\ldots,v_n$ iff
>
> (i) there is a substitution for $v_1,\ldots,v_n$ that makes each proposition in Γ true;
>
> (ii) every substitution for $v_1,\ldots,v_n$ that makes each proposition in Γ true, makes ψ true.

Clearly, Bolzanian deducibility resembles the Tarskian notion of logical consequence in several respects.[22] At this point, we wish to stress two important differences between both notions. A first difference results from clause (i) in the definition, which, roughly speaking, ensures that nothing is deducible from incompatible or inconsistent premises. Secondly, and more importantly for what follows, deducibility is a *ternary* relation. It obtains between a collection of premises, a conclusion, *and* a collection of ideas that are considered to be variable. A result of this is that there are arguments that are deductively valid with respect to some collection of ideas they contain, although they are not logically valid. The proposition [Socrates is mortal] is, for instance, deducible from [Socrates is human] with respect to [Socrates], even though the premise does not logically

[18] For a recent discussion of Bolzano's method of variation, see Morscher, "Bernard Bolzano," sect. 4.6. In order to prevent some technical problems, one has to introduce a number of constraints as to which substitutions are admissible. On this cf. Morscher, "Method of Variation."

[19] We use lower-case Greek letters as variables ranging over individual Bolzanian propositions and upper-case Greek letters as variables ranging over collections of Bolzanian propositions. As variables ranging over ideas we use v, v_1, v_2, … w, w_1, w_2, …. To give a concrete example, the proposition [Kant is a philosopher] is a *variant* of [Locke is a philosopher] with respect to the idea [Locke].

[20] We use the term 'collection' as a translation of Bolzano's term '*Inbegriff*.' By the latter Bolzano signifies several kinds of mereological compounds, i.e. objects that have *parts*; cf. *WL*, §§82–84. For present purposes, it is not necessary to go into Bolzano's mereology—apart from mentioning the fact that there is no such thing as an empty collection in Bolzano's universe. For more on Bolzanian collections, consider Betti, "Bolzano's Universe," and Krickel, *Teil und Inbegriff*.

[21] See *WL*, §155. We have somewhat simplified Bolzano's definition of deducibility. Bolzano himself defines deducibility for a collection of conclusions rather than a single conclusion.

[22] Cf. Tarski, "Folgerung." For a detailed comparison, consider Rusnock and Burke, "Bolzano and Etchemendy"; Siebel, "Consequence"; and Siebel, *Ableitbarkeit*.

entail the conclusion. By invoking a distinction between logical and non-logical ideas one can, however, easily obtain a binary relation of *logical* deducibility along the following lines: a proposition ψ is logically deducible from Γ iff ψ is deducible from Γ with respect to all non-logical ideas contained in the propositions in Γ and ψ.[23]

2. GROUNDING

Against the background of the previous section, we can now provide a brief overview of the basic relational properties of Bolzanian grounding.[24]

Terminology

Let us first introduce some terminology and some distinctions by considering a concrete example. In the following argument, the two premises stand in the relation of grounding to the conclusion, according to Bolzano.[25]

Socrates is a philosopher.
Socrates is an Athenian.

Socrates is a philosopher and an Athenian.

Bolzano calls the collection of premises (i) and (ii) the *complete ground* of (iii).[26] Each of (i) and (ii) is a *partial ground* of the conclusion. We will also say that (i) and (ii) stand in the relation of *partial grounding* to (iii). The conclusion (iii), in turn, is called the *consequence*. In fact, the conclusion in this example is only a *partial* consequence of its complete ground, as we will see.

Basic Properties of Grounding

The most important relational properties of Bolzano's notion of grounding can nicely be introduced by contrasting grounding with deducibility. To begin with, it is clear that the conclusion in our example is deducible from its complete ground with respect to [Socrates], [Athenian], and [philosopher]. This is even a case of logical deducibility.[27] Grounding and deducibility are thus compatible. Bolzano is not sure, though, whether grounding always goes in hand with deducibility. In other words, there might be truths that are not deducible from their complete

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[23]Bolzano suggests such a definition in *WL*, §223 [II.392], and Bolzano, "Lehrart," §8.3. On Bolzano's distinction between logical and non-logical ideas, see Rusnock and Burke, "Bolzano and Etchemendy," and Künne, "Analyticity and Logical Truth," 260.

[24]The key passages on Bolzano's notion of grounding in the *WL* are §§162, 168, 198–222. For more detailed discussions of the basic relational properties of grounding, cf. Berg, *Bolzano's Logic*, ch. VI; Berg, *Einleitung*; Tatzel, "Ground and Consequence," §§6–7; Tatzel, "Bolzano on Grounding"; Centrone, "Begründungen," §3; Centrone, *Early Husserl*, 125; Rumberg, "Normal Proofs," §3; and Schnieder, "Causation," 311–15.

[25]The example occurs in *WL*, §199 [II.344].

[26]When Bolzano writes 'the ground of ψ,' he usually means the *complete ground* of ψ; cf. *WL*, §168 [II.207].

[27]Bolzano discusses the respective argument scheme in a section of the *WL* that is devoted to "those modes of deduction whose correctness can be shown from logical concepts alone" (*WL*, §223 [II.392]); cf. also *WL*, §227 [II.411].

ground.[28] It is clear, in any case, that deducibility is not sufficient for grounding. First, grounding only obtains among *true* propositions, whereas deducibility can also obtain among false ones.[29] Second, deducibility can obtain mutually. This is the case in the above example. The conclusion is deducible from the premises, and each of the premises is deducible from the conclusion—with respect to the very same collection of ideas. Grounding, on the other hand, is an asymmetric relation, according to Bolzano. No truth can be grounded in one of its consequences—not even partially:[30]

Asymmetry: If φ is a partial ground of ψ, then ψ is not a partial ground of φ.

Consequently, partial grounding is also irreflexive. The asymmetry of partial grounding captures a general intuition many philosophers have had, and still have, about explanations.[31] Explanations must not be circular. No truth can explain itself, nor truths that explain it.

A further important difference between grounding and deducibility consists in the fact that the very same conclusion is in general deducible from various different collections of premises. Not so for grounding: there are several passages in which Bolzano states that for any truth, there is *at most one* complete ground.[32]

Uniqueness: If Γ is the complete ground of ψ, then there is no Δ different from Γ such that Δ is the complete ground of ψ.

This claim clearly distinguishes Bolzano's account of grounding from modern accounts of metaphysical grounding.[33] It also does not straightforwardly correspond to some intuition about explanation that is uncontested among philosophers who think about explanation nowadays. Bolzano seems to assume that for every truth that can be explained at all, there is exactly one complete explanation.

Bolzano points out that the two properties introduced above entail that grounding is neither monotonic nor transitive. In fact, he points out that it is even anti-transitive in the following sense:[34]

Anti-Transitivity: If Δ is the complete ground of φ and $\{\varphi\}\cup\Gamma$ is the complete ground of ψ, then $\Delta\cup\Gamma$ is *not* the complete ground of ψ.

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[28]In *WL*, §200, Bolzano presents a single example of such a case. For a discussion of this example, consider Tatzel, "Bolzano on Grounding," 249–53; and Betti, "Explanation in Metaphysics," 297–300.

[29]See *WL*, §201.

[30]See *WL*, §209.

[31]Cf. Newton-Smith, "Explanation," 128.

[32]See *WL*, §213 [II.372]; §312 [III.230]; and §528 [IV.266]. In other passages, however, Bolzano contradicts this claim; cf. *WL*, §168 [II.208] and §206 [II.359–60]. What he commits himself to quite explicitly and unambiguously is the claim that there is at most one complete ground for any *complete* consequence; cf. *WL*, §206 [II.359]. For more on the topic, consider Rumberg, "Normal Proofs," 432; Buhl, *Ableitbarkeit und Abfolge*, 28–29; Roski, *Bolzano's Notion of Grounding*, 153–58; and Tatzel, "Ground and Consequence," 13.

[33]For, according to virtually all those accounts, both disjuncts of a disjunction taken separately equally count as a complete ground of the disjunction, in case both of them are true. Moreover, each true instance of an existential quantification counts as a complete ground of the latter; cf. Correia and Schnieder, "Opinionated Introduction," 17. Unfortunately, Bolzano nowhere specifies any reasons for why he takes grounds to be uniquely determined.

[34]See *WL*, §213.

A final property of grounding that deserves to be mentioned is the *seriality* of partial and complete grounding. According to Bolzano, *any* truth and *any* collection of truths figure as the complete ground of some other truths. The reason for this is that Bolzano accepts a claim that is central to many modern theories of grounding and that is often termed "Aristotle's Insight": for every truth φ, it holds that φ is the complete ground of [φ is true].[35] Bolzano even accepts the following generalization of this principle: for every collection of truths Γ, it holds that Γ is the complete ground of [Each proposition in Γ is true].[36] Among the partial consequences of (i) and (ii) in our example above would thus also be the truth [Each proposition in {(i),(ii)} is true]. As we shall see below, the seriality of grounding yields some tensions in Bolzano's theory.

Mediate Grounding and Fundamental Truths

Since grounding is anti-transitive, the grounds *of* the grounds of some truth ψ (as well as *their* respective grounds, etc.) are strictly speaking not grounds of ψ. Bolzano calls all truths that are either immediate or remote grounds of a given truth ψ *supporting truths* (*Hülfswahrheiten*) of ψ, or truths on which ψ *depends* (*abhängt*), or *mediate grounds* (*mittelbare Gründe*) of ψ.[37] Without ascribing to Bolzano anything he would not agree to, we may characterize a relation of *mediate grounding* as the transitive closure of partial grounding.[38] Bolzano states that mediate grounding is irreflexive, from which it follows by transitivity that it is also asymmetric and thus, in modern terminology, a strict partial order on the collection of all true propositions.

Bolzano assumes that at least some chains of mediate grounds terminate eventually. In other words, there are truths that do not have a ground. Bolzano calls those truths *fundamental truths* (*Grundwahrheiten*).[39] This idea can also be related to the notion of explanation: according to Bolzano, at least some explanations must ultimately come to an end.

3. SIMPLICITY AND ECONOMY

In the previous section, we have discussed the basic relational properties of Bolzanian grounding. One might interpret those properties as the minimal requirements an explanatory relation has to fulfill, according to Bolzano. However, none of the relational properties is very informative when we are in search of an explanatory argument for some given truth. With several claims that relate to what we have called (Simplicity) and (Economy) in the introduction, Bolzano introduces properties of grounding that are much more informative in this respect, as these properties take into account the internal buildup of the propositions involved.[40]

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[35]See *WL*, §205. On Aristotle's Insight, see e.g. Schnieder, "Truthmaking"; and, more generally, Künne, *Conceptions of Truth*, ch. 3.5.

[36]See *WL*, §205.

[37]See *WL*, §217.

[38]Cf. the discussion in *WL*, §§217–19.

[39]See *WL*, §214.

[40]For reasons of space, we cannot discuss a further principle of Bolzano's theory of grounding that also corresponds to a classical idea on explanation, namely that the grounds of a given truth must exhibit a maximal degree of *generality*; cf. *WL*, §221 [II.387]. The principle is discussed in Roski, *Bolzano's Notion of Grounding*, ch. 5.4.

One brief remark is in order here. In section 1, we mentioned the distinction between conceptual and intuitional truths. The part of Bolzano's theory that we discuss now is constrained to conceptual truths, that is, to truths of mathematics, metaphysics, and similar non-empirical disciplines. According to Bolzano, conceptual truths have the peculiarity that they can be grounded in other conceptual truths only: each partial ground of a conceptual truth is again a conceptual truth.[41]

Before we discuss Bolzano's claims concerning (Simplicity) and (Economy) in full detail, let us have a look at a passage that introduces the general idea rather clearly and helps to get an impression of what Bolzano is after. When concerned with the question how to discover the complete ground of a given conceptual truth, Bolzano writes the following:

> (B1) An analysis of the given proposition that extends to its simple parts, insofar as we are able to perform it, must be our first business in this problem. Next, we must construct from the parts we have discovered in the proposition *M* propositions that are simpler or at least not more complex than *M*, and also constituted so that *M* is deducible from them. If in doing this we find it necessary to have recourse to concepts not contained in *M*, we must seek to reduce their number as far as possible. Only when we have succeeded in this way in assuring ourselves that the truths *A*, *B*, *C*, *D*, … from which *M* is deducible, are each simpler or at least no more complex than *M*, and taken together constitute a simpler collection than any other from which *M* may be deduced may we permit ourselves to look upon the former as the ground of the latter.[42]

When in search of the complete ground of a given truth *M*, Bolzano instructs us to find truths from which *M* is deducible and which individually are "simpler or at least no more complex than *M*."[43] This condition corresponds to what we call (Simplicity). Bolzano further requires that the premises must "constitute a simpler collection than any other from which *M* may be deduced."[44] This condition corresponds to our (Economy). Since the condition concerns the complete ground of the truth in question, we call it *Complete Ground Economy*, in order to distinguish it from a related condition concerning partial grounds, which we accordingly call *Partial Ground Economy*. The passage under consideration also nicely illustrates the role of deducibility in the present context: deducibility provides the relevant reference class. From all deductively valid arguments for a given truth, the explanatory one is supposed to be singled out as being the simplest and most economical.

The properties of grounding that Bolzano alludes to in (B1) are revealing for his views on grounding as an explanatory relation. For it is traditionally a rather influential idea that the explanatory order proceeds from the simple to the more complex and that it is characterized by some kind of economy.[45] For Bolzano, such

[41] See *WL*, §221.1 [II.384].

[42] *WL*, §378 [III.496].

[43] *WL*, §378 [III.496].

[44] *WL*, §378 [III.496].

[45] Concerning the idea that the explanatory order proceeds from the simple to the more complex, see e.g. Betti and de Jong, "Classical Model of Science," 188. A nice expression of the view that explanatoriness goes in hand with economy by an author whose concerns are in many ways related to

ideas do not merely relate to pragmatic or epistemic constraints on explanations. Rather, they derive directly from objective characteristics of grounding. In the following sections, we discuss how Bolzano spells out these characteristics in detail.

Before we begin our discussion, we would like to add one methodological note. As will become explicit below, the passages in which Bolzano discusses the claims that appear in (B1) in more detail are often difficult to interpret. In particular, various claims of Bolzano turn out to be false when taken fully literally. In many cases, however, there are natural amendments or adjustments to what he writes, which we will incorporate in our interpretation. For this reason, our interpretation will often have the character of a *rational reconstruction* in the sense that we will suggest principles that are often idealized versions of what Bolzano literally writes. This way, the ideas that stand in the background of his claims can be pinned down more clearly, and features of his account that are genuinely problematic can more easily be distinguished from mere negligence or inaccuracy in his formulations.

3.1 Simplicity

In passage (B1), Bolzano alludes to a principle that concerns the complexity of the partial grounds of a given conceptual truth. According to him, the grounds of a conceptual truth are never more complex than their consequence. Elsewhere, Bolzano spells out this idea in a bit more detail as follows:

> (B2) I believe that a purely conceptual truth upon which a second one depends must never be more complex than the latter, though it does not have to be simpler. Propositions which form the objective ground of a purely conceptual truth must separately contain no more simple parts than the truth which depends upon them.[46]

Bolzano here demands that the mediate grounds of a conceptual truth—and thus, in particular, also its partial immediate grounds—can contain at most as many *simple parts* as the corresponding consequence.

We have mentioned earlier (cf. section 1) that Bolzanian propositions are built up from ideas. Their ultimate constituents are simple ideas that are not further decomposable. Bolzano explicitly acknowledges the possibility that a simple idea occurs more than once in a given proposition.[47] If we decompose a proposition step by step, it may therefore happen that we come across the very same simple

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that of Bolzano is the following remark: "Indeed the essence of explanation lies precisely in the fact that a wide, possibly unsurveyable manifold is governed by one or a few sentences" (Frege, *NS*, 40). The idea that theories and proofs that reflect the explanatory order exhibit a certain kind of economy can of course also be found in many historical texts on proper method. Aristotle writes in *Post. An.* 86a that "one demonstration [is] better than another if, other things being equal, it depends on fewer postulates, or suppositions, or propositions." The role of an analogous assumption in Euclid's edification of the *Elements* is highlighted by Heath in his commentary (*Elements*, I.124). Consider, moreover, Leibniz's remarks on geometrical rigor (cf. Leibniz, *NE* IV, ch. VII, §6), as well as Arnauld and Nicole, *Art of Thinking*, 256; Newton, *The Principia*, 794; and Kant, *CPR* B680–81. Modern defenders of this idea are adherents of the so-called *unificatory* theory of scientific explanation, such as Friedman, "Scientific Understanding," and Kitcher, "Explanatory Unification."

[46] *WL*, §221.2 [II.384].

[47] For a discussion of the question how an idea as an abstract object can have multiple occurrences within a proposition, cf. Künne, "Propositions," 223–26.

idea several times. For illustration, consider the proposition [Peter is not rich and not happy], which contains two occurrences of the idea [not]. Bolzano takes it that this proposition is more complex than the proposition [Peter is rich and not happy], which contains the idea [not] only once.[48] It is thus important to count all *occurrences* of simple ideas—rather than only the simple ideas themselves—when concerned with the complexity of a proposition. The "simple parts" that Bolzano mentions in (B2) must accordingly be understood as *occurrences of simple ideas*.[49] The number of occurrences of simple ideas into which a given proposition can be decomposed determines its *degree of complexity*. A proposition is more complex than another if it contains more occurrences of simple ideas and hence is of a higher degree of complexity than the latter. Against this background, we can formulate the principle Bolzano states in (B2)—which we call the *Complexity Constraint*—as follows:[50]

> *Complexity Constraint*: If φ is a partial ground of a conceptual truth ψ, then the degree of complexity of φ does not exceed the degree of complexity of ψ.

The Complexity Constraint entails that no conceptual truth can be explained by truths that contain more occurrences of simple ideas than the truth they are supposed to explain.

Thus stated, the principle only puts a restriction on the *number* of occurrences of simple ideas from which the partial grounds of a conceptual truth can be built up. It does not put any restriction on what *kind* of simple ideas can be contained within those partial grounds. In several passages of his writings, however, Bolzano seems to require more.[51] What he seems to have in mind is that the truths by which a given conceptual truth is explained should be composed of only those simple ideas that occur in the given truth as well. We find this idea back in the quote (B1) above. When in search of the grounds of a given conceptual truth *M*, Bolzano instructs us to first decompose the given truth *M* step by step into its simple ideas and then "construct from the parts we have discovered in the proposition *M* propositions that are simpler or at least not more complex than *M*,"[52] and from which *M* is deducible. Only if absolutely necessary should we make use of ideas not occurring in *M*.

This suggests that Bolzano assumes that the partial grounds of a conceptual truth are—at least in a majority of cases—built up from only those simple ideas into which the consequence can be decomposed. At least in general, the partial grounds of a conceptual truth do not contain any occurrence of a simple idea that does not occur in the consequence, and they do not contain more occurrences of the very same simple idea than the consequence does. Due to this claim, Bolzano's concept of grounding has frequently been compared to current notions of normal

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[48]Bolzano explicitly discusses a related example in *WL*, §32 [I.147].

[49]For a detailed argument of the claim that what has to be taken into account here are occurrences of simple ideas rather than the simple ideas themselves, see Tatzel, "Proving and Grounding," sect. 2.3, and Roski, *Bolzano's Notion of Grounding*, 201–4.

[50]We adopt the name from Tatzel, "Proving and Grounding."

[51]Cf. Bolzano, "Lehrart," §17; *WL*, §378 [III.496]; and §609 [IV.449]. Cf. also the early Bolzano, "Parallelen," 137.

[52]*WL*, §378 [III.496].

proofs, where the claim finds its analogue in the subformula principle.[53] The claim also relates to a classical ideal of *purity* of explanatory proofs, namely the idea that an explanatory proof for a given proposition should not make use of concepts *alien* to that proposition.[54] It is an open question why exactly Bolzano could not exclude the possibility that in some cases the partial grounds do contain ideas that do not occur in the respective consequence after all. In the context of this paper, we have to leave this question open.[55]

Consider the following example:

The natural numbers are well-ordered.
The natural numbers are closed under addition.

The natural numbers are well-ordered and closed under addition.

Although those propositions are not yet analyzed into their simple parts, it is clear that the number of occurrences of simple ideas into which each of the partial grounds can be decomposed is smaller than the number of occurrences of simple ideas from which the consequence is built up. And not only are the partial grounds individually not more complex than their consequence, but each of them contains only occurrences of simple ideas that occur in the consequence as well.

3.2 *Economy*

We now turn to two claims of Bolzano that relate to (Economy). We begin with a claim that has already been alluded to in (B1) and that requires that complete grounds must exhibit a certain kind of economy.

3.2.1 *Complete Ground Economy*

In passage (B1), Bolzano indicates that for a collection of truths from which a given truth *M* is deducible to constitute *M*'s complete ground, the truths in question must "taken together constitute a simpler collection than any other from which *M* may be deduced."[56] Elsewhere in the *WL*, Bolzano states this principle a bit more precisely and makes two important provisos explicit (marked by '(α)' and '(β)' in the quote below). These provisos restrict the comparison class relative to which the respective complete ground is the simplest collection of premises:

> (B3) [T]he truths *A*, *B*, *C*, … constituting the ground of a truth *M*, which is also deducible from them, must always be the simplest collection of truths from which *M* is deducible, provided that [(α)] always the same ideas are considered variable, and that [(β)] none of the premises are individually more complex than the conclusion.[57]

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[53]For a detailed comparison of Bolzano's concept of grounding with cut-free proofs in Gentzen's sequent calculus, see Tatzel, "Proving and Grounding," and Rumberg, "Normal Proofs." Cf. also Centrone, *Early Husserl*, 126–27; and Centrone, "Begründungen," 22. In Rumberg, "Normal Proofs," the discussion is extended to normal natural deduction proofs and their role in proof-theoretic semantics.

[54]Cf. Detlefsen, "Purity." Centrone ("Strenge Beweise") has shown that this ideal plays an important role in Bolzano's early theory of explanatory proofs.

[55]In Roski, "Bolzano's Notion of Grounding," 210–11, it is conjectured that this might be related to Bolzano's views on the role of syllogisms in explanatory proofs.

[56]*WL*, §378 [III.496].

[57]*WL*, §221.5 [II.387]. Note that Bolzano does not *explicitly* restrict this claim (and various related ones we shall discuss below) to conceptual truths, but the context strongly suggests such a restriction.

What Bolzano seems to have in mind is the following: a deductively valid argument for a given truth ψ from a collection of premises Γ with respect to a collection of variable ideas v is explanatory *only if* there is no smaller collection of premises from which ψ is likewise deducible [(α)] with respect to v [(β)] and which does not contain any premise that is of a higher degree of complexity than ψ (or identical to ψ).

A few comments on our construal of (B3) are in order. First, that no premise for ψ must be identical to ψ is not mentioned in (B3). However, without this restriction, the principle would be false for the trivial reason that every truth is deducible from itself and thus constitutes in many cases a smaller collection of premises than its complete ground. Second, note that we assume that by the "simplest collection" Bolzano means the collection with the fewest proximate parts, that is, the collection containing the fewest propositions.[58] This leads us to another concern. In our reconstruction of (B3), we have not taken over Bolzano's use of the definite description and the superlative ("the simplest"). For it is generally not the case that there is always a unique smallest collection of premises from which a given truth is deducible under the given provisos.[59] Since we assume that Bolzano was aware of that fact, we read him charitably as claiming that there is no collection of premises from which the given truth is deducible under the given provisos that contains *fewer* truths than its complete ground, even though there might be other equally small collections of premises. The complete ground is supposed to be a *minimal* collection.

Let us now take a closer look at the two provisos that Bolzano introduces in (B3). Proviso (β) restricts the comparison class to all those collections of premises that satisfy the Complexity Constraint (cf. section 3.1). Without this restriction, the complete ground of a given truth would only rarely constitute a minimal collection of premises for that truth. For by conjoining all partial grounds of some truth ψ into a single truth (for instance by forming a conjunction), one can easily obtain a smaller collection of premises from which ψ is still deducible.[60]

[58]One might think that Bolzano means by 'the simplest collection of truths' not only the smallest collection, but rather the collection that contains the fewest proximate and remote parts, i.e. both the fewest propositions and the fewest parts of propositions. Two reasons speak against such a reading. First, on this reading, another of Bolzano's economy principles that concerns the parts of the propositions contained in a complete ground and that we shall discuss below would become superfluous. Second, when Bolzano speaks of the *parts of a collection of F*s without qualification, he usually only means the parts of the collection that are themselves *F*s; cf. his remarks in *WL*, §86 [I.407]. In particular, the simplest collection of *F*s is the collection that contains the fewest *F*s. Accordingly, the simplest collection of *truths* should be the collection that contains the fewest *truths* (rather than the fewest parts of truths).

[59]To see this, assume, for instance, that a truth ψ is deducible from its complete ground and one of the partial grounds is of the form [*A*s are not-B]. In this case ψ will also be deducible from an equally small collection of truths that contains [*B*s are not-*A*] instead of [*A*s are not-*B*]. Both propositions, however, count as different propositions given Bolzano's fine-grained identity-criterion for propositions; cf. Morscher, "Bernard Bolzano," sect. 3.4.

[60]This echoes a related problem for the unificatory theory of scientific explanation that Michael Friedman has once called "the problem of conjunctive trivialization" (Friedman, "Scientific Understanding," 18).

However—thus, presumably, Bolzano's idea—this single truth will be of a greater degree of complexity than ψ and is therefore excluded from the comparison class in virtue of (β).[61]

Proviso (α) deserves a bit more attention. This proviso requires that the collection of variable ideas with respect to which a given truth is deducible from its complete ground has to remain the same when comparing the complete ground to other possible collections of premises. Such a requirement is necessary due to the fact that deducibility is a ternary relation: the more ideas one varies, the more premises one might need. For concreteness, consider again the example that we have mentioned earlier.

The natural numbers are well-ordered.
The natural numbers are closed under addition.

The natural numbers are well-ordered and closed under addition.

Whereas in this example the number of premises is arguably minimal if the conclusion is deduced with respect to the ideas [the natural numbers], [well-ordered], and [closed under addition], the premises do *not* constitute a minimal collection when we take only the idea [well-ordered] to be variable. If we vary only the idea [well-ordered], the second premise is superfluous: no substitution for [well-ordered] that makes the first premise true can make the conclusion false. However, it is clear that Bolzano takes the complete ground of the conclusion to consist of *both* premises.[62]

This observation shows that (B3) cannot hold as it stands. For nothing of what is stated in (B3) allows us to decide relative to which collection of ideas deducibility is supposed to hold. Even if one keeps the collection of variable ideas fixed when comparing the complete ground of a given truth to other possible collections of premises (as [α] demands), one has to make a choice as to which ideas are considered variable in the first place.

Hence, in order for Bolzano's principle to be feasible, one needs to restrict the collection of variable ideas uniformly. In what follows, we will assume that deducibility holds with respect to all non-logical ideas. In other words, we suggest a reformulation of the principle proposed in (B3) in which the ternary relation of deducibility is replaced by the binary relation of logical deducibility. This is not just an adhoc restriction motivated to avoid the particular counterexample we have discussed. Rather, it is suggested by many examples for cases of grounding-*cum*-deducibility that Bolzano discusses.[63] In those cases, Bolzano seems to take as many (non-logical) ideas as possible to be variable, or, in other words, considers

[61]Without further assumptions, this is of course insufficient to avoid the problem. For what guarantees that conjoining all partial grounds of a given truth ψ *always* results in a truth more complex than ψ? In what follows, we will, however, grant Bolzano the point.

[62]Cf. *WL*, §199; §221.7 [II.388]; and §227 [II.411].

[63]Cf. e.g. *WL*, §198 [II.339–40]; §199 [II.344]; §221.7 [II.388]; §225 [II.399]; §226 [II.407]; and §227 [II.411]. For an extended argument, as well as other examples, see Roski, *Bolzano's Notion of Grounding*, ch. 5.3.1.

argument patterns as general as possible.[64] Note that little of what follows hinges on our restriction to logical deducibility. *Some* restriction needs to be made for Bolzano's idea to be feasible, and the present one fits the thrust of many of his remarks.

At this point, we can summarize our reconstruction of what Bolzano writes in (B3):

> *Complete Ground Economy*: If Γ is the complete ground of ψ and ψ is logically deducible from Γ, then there is no smaller collection of premises Γ' (not containing ψ) such that ψ is logically deducible from Γ' and no proposition in Γ' is of a higher degree of complexity than ψ.

To put it in more intuitive terms: in order to decide whether a given logically valid argument for some truth ψ is explanatory, one has to determine whether there is a logically valid argument for ψ with fewer premises none of which is more complex than the conclusion. If the given argument does not pass this test, it contains more premises to logically deduce ψ than strictly necessary and thus—so Bolzano's idea—it will contain explanatorily irrelevant truths.

Complete Ground Economy has some interesting corollaries but also reveals a certain tension in Bolzano's theory of grounding. A first corollary is to the effect that if a truth is logically deducible from its complete ground, the respective partial grounds are logically independent of each other in the following sense: no truth that is part of such a complete ground is logically deducible from the remainder.[65] For otherwise, the truth in question would clearly be a redundant premise in a logically valid argument. A second corollary is to the effect that partial grounds cannot be logically analytic. A logically analytic truth in Bolzano's sense is a truth that remains true under all substitutions of its non-logical constituents.[66] However, if a premise in a logically valid argument remains true under *all* substitutions of its non-logical constituents, that premise does not at all contribute to the validity of the argument. It can be deleted, and the conclusion is still logically deducible from the remaining premises. For a related reason, logically analytic truths cannot occur as consequences. For if the conclusion of an argument remains true under all substitutions of its non-logical constituents, *none* of the premises contributes to the validity of the argument.[67]

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[64]We wish to thank Arianna Betti for pointing out the connection to the notion of generality in this context.

[65]As a referee has pointed out to us, the condition does not exclude that there are other kinds of logical dependence among partial grounds. If φ and ψ are partial grounds of some given truth, then some logical consequences of ψ may, for instance, just as well be logical consequences of φ (i.e. the partial grounds may have some logical consequences in common).

[66]For a recent discussion of Bolzano's notion of analyticity cf. Rusnock, "Analyticity." Note that logically analytic truths are not to be confused with truths that belong to the science of logic. The latter may very well be synthetic according to Bolzano; cf. Morscher, "Logisch Wahr."

[67]Bolzano noticed closely related facts elsewhere. When he discusses a special case of deducibility, called "exact deducibility," which does not allow for redundant premises, he points out that neither premises nor conclusions of cases of exact deducibility may be analytic; see *WL*, §155.27 [II.124]; and Rumberg, "Normal Proofs," 444, n. 53. Note that the above corollaries might not hold in full generality. Since, according to Bolzano, there cannot be cases of deducibility with zero premises, Complete Ground Economy might not exclude the possibility that if a complete ground consists of only a single proposition, this proposition may be logically analytic, or that the consequences of such a ground may be logically analytic; cf. Roski, "Bolzano's Notion of Grounding," 235–36. For present purposes, however, we do not need to go into these further complications.

There are a few passages in Bolzano's writings that seem to cohere with these corollaries—especially with the second one. Bolzano argues, for instance, that logically analytic truths are "too unimportant to merit presentation as one of the propositions of a science"[68] and that they "do not even contain the partial ground of any [noteworthy] truth."[69] That logically analytic truths never contribute to the ground of any other truth, as Complete Ground Economy entails, would explain why they are insignificant in the way Bolzano indicates here.

Yet, the corollaries reveal a severe tension with another idea of Bolzano that we have mentioned previously (cf. section 2). Recall that Bolzano accepts a generalization of Aristotle's Insight: *every* collection of truths Γ is the complete ground of [Each proposition in Γ is true]. This, however, cannot be the case if Complete Ground Economy holds. In particular, if Complete Ground Economy holds, a collection of truths Γ that contains truths that are not logically independent or logically analytic can never figure as the complete ground of any truth.

Here, two general ideas of Bolzano seem to clash. On the one hand, there is the idea that grounding is, as it were, a very inclusive notion: *every truth* and *every collection of truths* is the complete ground of some other truths. On the other hand, there seems to be the idea that grounding is a very exclusive notion: only *distinguished* collections of truths—those that do, for instance, not contain logically analytic truths—form the complete grounds of other truths. When these ideas are spelled out the way Bolzano does in the *WL*, they seem to be irreconcilable. While this does not exclude that there might be other ways to spell out these ideas that do not give rise to the tension we have indicated, we do not see any straightforward way to do so.

3.2.2 *Partial Ground Economy*

Next to the above discussed principle, which captures the idea that *complete* grounds exhibit a certain kind of economy, Bolzano also introduces a principle that can be read as capturing the idea that *partial* grounds exhibit a certain kind of economy. Also in this case, Bolzano makes use of the notion of deducibility, in particular of the notion of mutual deducibility or, in his terminology, *equivalence* (*Gleichgültigkeit*).[70] The principle is stated as follows:

> (B4) [E]ach of the truths *A, B, C, D*, … , which together form the ground of truth *M*, is always the simplest of the propositions that are individually equivalent to it.[71]

The partial grounds of a given truth are distinguished by having the lowest degree of complexity among all equivalent truths. Since the relation of equivalence inherits the ternary character of the relation of deducibility, the interpretation of (B4) gives rise to problems quite similar to the ones we have encountered in the interpretation

[68] *WL*, §12 [I.52]. In this passage, Bolzano explicitly rejects the view that truths that belong to *logic* are analytic.

[69] *WL*, §45 [I.203]. That analytic truths play a subordinate role in proper sciences according to Bolzano has been conjectured by de Jong, "Analyticity," 348; cf. also de Jong, "The Analytic-Synthetic Distinction," 259–60. Bolzano held similar views already in his early work; cf. Bolzano, *Beyträge*, 81.

[70] The notion of equivalence in Bolzano's sense is defined in *WL*, §156.

[71] *WL*, §221.4 [II.386].

of (B3). In particular, one has to settle on some uniform way to decide with respect to which ideas equivalence is supposed to hold. As in our discussion above, our way to deal with this problem is to substitute the ternary relation of equivalence with the binary relation of logical equivalence. Furthermore, also in this case, we suggest that one should charitably read Bolzano's phrase 'the simplest' as *among the simplest*, as there might not be a unique simplest truth in each case. Against this background, we offer the following reconstruction of Bolzano's claim in (B4):

> *Partial Ground Economy*: If φ is a partial ground of ψ, then there is no truth logically equivalent to φ that is of a lower degree of complexity than φ.

One might put the idea that comes to the fore in this principle as follows: truths that are logically equivalent to a given truth φ but are more complex than φ will contain constituents that are explanatorily irrelevant. Intuitively, those constituents do not contribute in any essential way to what the truth *says* or *is about*, but just vacuously increase its complexity. While this idea undoubtedly has a certain initial plausibility for an explanatory relation, Bolzano's way to make it precise yields severe tensions with other parts of his theory.

Immediately after Partial Ground Economy Bolzano states an analogous principle for partial consequences. Passage (B4) continues as follows:

> (B5) [E]very truth which must be considered a consequence of the several truths, *A*, *B*, *C*, … , is always the simplest of all propositions individually equivalent to it.[72]

A consequence of Partial Ground Economy and of what Bolzano writes in (B5) is that there can be no grounding relations among logically equivalent truths of different degrees of complexity.[73] This, however, stands in sharp contrast to various passages in the *WL* in which Bolzano states that there *are* grounding relations among logically equivalent truths of different degrees of complexity.[74] In particular, it directly contradicts Aristotle's Insight, as Bolzano assumes that truths of the form φ and [φ is true] are logically equivalent.[75] More strikingly, it contradicts what Bolzano writes immediately after the passages (B4) and (B5), namely:

> (B6) Thus it follows that all truths which are equivalent to a given individual truth *M*, and which are more complex than it, are always consequences of this truth alone, etc.[76]

By what Bolzano writes in (B5), however, no truth can have a logically equivalent truth of a higher degree of complexity as a consequence, since such a truth would obviously not be the simplest among logically equivalent ones. Something has gone wrong here. It seems clear that Bolzano assumes that grounding imposes a certain fine-grained hierarchy among logically equivalent truths. Two cases in point would be the following:

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[72] *WL*, §221.4 [II.386].

[73] Also here, we read 'equivalent' as *logically equivalent*.

[74] Cf. *WL*, §45 [I.206] and §209 [II.363–64].

[75] For [true] is a logical notion according to Bolzano; cf. e.g. Künne, "Analyticity and Logical Truth," 260.

[76] *WL*, §221.4 [II.386].

ψ	ψ
ψ is true	$\neg\neg\psi$
[ψ is true] is true	$\neg\neg\neg\neg\psi$
[[ψ is true] is true] is true	$\neg\neg\neg\neg\neg\neg\psi$
$\vdots$	$\vdots$

Given what Bolzano says about the grounds of doubly negated propositions, and given that he accepts Aristotle's Insight, each truth in such a hierarchy should be the immediate partial ground of the next one.[77] However, while all truths in such a hierarchy are logically equivalent to each other, each of them is of a different degree of complexity. Hence, neither are all partial grounds nor all partial consequences in the hierarchy of a lower degree of complexity than all truths logically equivalent to them. The latter holds only for the respective simplest truths in the hierarchy.

While Partial Ground Economy does not seem to be tenable as a necessary condition for grounding, it thus at least suggests a plausible necessary condition for a truth being *fundamental*: if a truth is fundamental, there is no logically equivalent truth of a lower degree of complexity.[78] Whether Bolzano's initial suggestions for necessary conditions in (B4)–(B6) can be modified so as to prevent the problems we sketched in this section, remains an open question that we have to leave open for further research.

Let us wrap up. In this section, we have considered a number of necessary conditions for grounding that take into account the internal structure of grounds and consequences: the Complexity Constraint, Complete Ground Economy, and Partial Ground Economy. We have shown that these constraints can be seen as motivated by classical views on explanation, and we have discussed a number of problems that appear when making them precise. In the remainder of the paper, we will look at another, more promising, role the constraints play in Bolzano's theory, namely their role as part of a proposal for a sufficient condition for grounding.

4. A SUFFICIENT CONDITION

Let us return to the question we started out with in the introduction: under what conditions is a deductively valid argument explanatory? The necessary conditions discussed in the previous section as such provide only *negative* criteria to answer this question. As the following passage shows, however, Bolzano takes those necessary conditions to be so central for his notion of grounding that he considers the possibility that they are jointly sufficient:

> (B7) [(a)] If a proposition *M* stands to other propositions *A*, *B*, *C*, … in the relation of *exact* deducibility (cf. §155, no. 26) with respect to ideas *i*, *j*, … , [(b)] if, moreover, propositions *A*, *B*, *C*, … and *M* are the simplest propositions among those equivalent to them, and [(c)] if none of *A*, *B*, *C*, … is more complex than *M*, then we may assume that *M* stands to *A*, *B*, *C*, … in a true relation of grounding, [(d)] in such a

[77]Cf. *WL*, §45, on the grounds of double negations, and *WL*, §205, on Aristotle's Insight. In this respect, Bolzano agrees with many participants of the current debate on metaphysical grounding; cf. Correia and Schnieder, "Opinionated Introduction," 11 and 17.

[78]We owe this last observation to Benjamin Schnieder.

way that whenever i, j, … are replaced by ideas which make propositions A, B, C, … not only true, but keep them from becoming redundant, then truth M is a proper consequence of truths A, B, C, ….[79]

In this passage, Bolzano specifies three conditions ([a]–[c]) that are supposed to ensure that the premises of a deductively valid argument stand in the relation of grounding to the conclusion. The conditions Bolzano mentions closely correspond to the necessary conditions that we have discussed above (cf. section 3). In the passage marked by '(d),' Bolzano even generalizes his claim to the effect that grounding thus characterized is preserved under variation, in case the admissible substitutions are restricted in a suitable way. Let us first have a closer look at the three conditions that taken together are supposed to be sufficient for grounding. In a further step, we will then investigate Bolzano's claim that under suitable conditions, grounding is retained under variation.

4.1 *Simplicity and Economy as Sufficient Condition*

Of the three conditions that Bolzano mentions in (B7), the conditions (b) and (c) are already familiar to us. Condition (b) corresponds to Partial Ground Economy (cf. section 3.2.2) and the constraint mentioned in (B5), while condition (c) corresponds to the Complexity Constraint (cf. section 3.1). Each premise must be of a lower degree of complexity than the conclusion, and both the premises and the conclusion must individually be among the simplest equivalent propositions. The general idea of Bolzano's proposal, it seems, is to gather enough necessary conditions for grounding that taken together are also sufficient. Against this background, it is surprising that no perfect analogue of Complete Ground Economy (cf. section 3.2.1) appears in the proposal as stated in (B7). Rather than requiring the premises to form a minimal collection, Bolzano merely requires them to stand in the relation of *exact deducibility* to the conclusion (cf. [a] in [B7]). A proposition is exactly deducible from a collection of premises in Bolzano's sense just in case none of the premises, not even any of their parts, can be omitted if deducibility is still to hold.[80] Although exact deducibility assures that none of the premises is redundant for deducing the required conclusion, it does not guarantee that the number of premises is minimal in this respect. As Bolzano was aware, a proposition can be exactly deducible from various collections of propositions of different cardinality.[81] The following example illustrates the point:

(i) The natural numbers are well-ordered.
(ii) The natural numbers are closed under addition.
(iii) The natural numbers are closed under multiplication.

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[79] *WL*, §221.7 [II.387–88].

[80] See *WL*, §155.26 [II.123]. As has frequently been pointed out, Bolzano's notion of exact deducibility captures at least some aspects of a relevant consequence relation; cf. George, "Relevance"; Siebel, *Ableitbarkeit*, ch. 6; and Stelzner, "Compatibility."

[81] Cf. *WL*, §262 [II.541].

(iv) The natural numbers are well-ordered and closed under addition and closed under multiplication.

(i') The natural numbers are well-ordered and closed under addition.
(iii) The natural numbers are closed under multiplication.

(iv) The natural numbers are well-ordered and closed under addition and closed under multiplication.

In this case (iv) is exactly deducible from (i), (ii), and (iii) with respect to the ideas [the natural numbers], [well-ordered], [closed under addition], and [closed under multiplication]. It is, however, likewise exactly deducible from only two premises with respect to the same collection of variable ideas, namely from (i') and (iii). In each of these arguments, none of the premises, nor any of their parts, is redundant. We thus have a case in which the very same proposition is exactly deducible from two collections of premises of different cardinality with respect to the very same collection of variable ideas. Note that in each case, the premises are at most as complex as the conclusion, so the Complexity Constraint is fulfilled as well. Given Complete Ground Economy, however, the argument resting on three premises is ruled out as an instance of grounding, as it makes use of more premises than necessary.[82] Exact deducibility is thus too weak a requirement in the sufficient condition, as it allows for cases that are in conflict with Bolzano's necessary conditions for grounding.

Given that Bolzano's general idea in (B7) seems to be to gather necessary conditions for grounding that are jointly sufficient, we suggest modifying his proposal accordingly. In particular, we suggest replacing the requirement that the conclusion must be exactly deducible from the premises by a requirement that corresponds to Complete Ground Economy: the premises must form a minimal collection from which the conclusion is deducible in such a way that the Complexity Constraint remains fulfilled. This modification seems to be very much in the spirit of Bolzano's original proposal and prevents the difficulty discussed above.

This brings us to a related point. We have seen earlier (cf. section 3.2.1) that the number of premises that is needed in order to deduce a given proposition depends on which ideas are considered variable. The more ideas are considered variable, the more premises might be needed. As it is formulated in (B7), Bolzano's proposal for a sufficient condition relies on the *ternary* relation of deducibility. Consequently, whether the condition, as it stands, is fulfilled in a given case depends on which ideas are taken to be variable. In other words, if Bolzano's condition is taken literally, what would count as ground and consequence would depend on the respective collection of variable ideas. In particular, the very same truth could have various complete grounds of different cardinality relative to different collections of variable ideas. When discussing Bolzano's necessary conditions above (cf. sections 3.2.1 and 3.2.2), we restricted his constraints to the case of logical deducibility and logical equivalence for related reasons, and we suggest

[82]Note that our argument depends on an assumption we made earlier, namely that Complete Ground Economy sets an upper bound on the *number of premises* in an explanatory argument rather than on the total number of *parts* of which these premises are composed; cf. n. 51 above.

doing the same here. This said, we suggest the following as a sufficient condition for Bolzanian grounding:

Sufficient Condition: A collection of truths Γ is the complete ground of a truth ψ if

(a) no truth in Γ is of a higher degree of complexity than ψ (Complexity Constraint);

(b) there is no smaller collection of truths than Γ from which ψ is logically deducible in a way that satisfies (a) and which does not contain ψ (Complete Ground Economy);

(c) for neither any truth in Γ nor ψ there is a logically equivalent one of a lower degree of complexity (Partial Ground Economy and [B5]).

Note that our rendering of Bolzano's idea contains a couple of further amendments. First, we made explicit that the premises and the conclusion must be true propositions, since grounding cannot obtain among false ones. Second, as grounding is asymmetric and consequently also irreflexive (cf. section 2), we demand that the conclusion must not be among the premises (as we did previously in our formulation of Complete Ground Economy; cf. section 3.2.1 above). Third, as in the case of the corresponding necessary conditions, we only require that the premises be *among* the smallest collections and that each premise and the conclusion should individually be *among* the simplest equivalent ones, rather than demanding that the premises must be *the* smallest collection and that each premise and the conclusion must be *the* simplest among equivalent ones (cf. sections 3.2.1 and 3.2.2). In order to see what this amounts to in the present context, consider the following example:

(i) The natural numbers are well-ordered and closed under addition.
(ii) The natural numbers are closed under multiplication.

(iii) The natural numbers are well-ordered and closed under addition and closed under multiplication.

(i') The natural numbers are well-ordered.
(ii') The natural numbers are closed under addition and closed under multiplication.

(iii) The natural numbers are well-ordered and closed under addition and closed under multiplication.

Both these arguments rely on a minimal collection of premises from which the conclusion is logically deducible, none of which is of a higher degree of complexity than the conclusion. If it were required that the collection of premises must be *the* smallest one, the sufficient condition would remain silent whether the arguments in question are instances of grounding. More generally, all cases in which there is no such thing as *the* smallest collection of premises (or *the* simplest among equivalent propositions) would have to be left undecided.[83] If, on the other

[83]A referee pointed out that merely sufficient conditions generally leave cases undecided. While this is correct, it still seems an unwelcome consequence that the condition would not even allow one to decide between the above two examples, given that Bolzano explicitly presents examples of this kind in order to illustrate the sufficient condition; cf. *WL*, §221 [II.388].

hand, it is merely required that the collection of premises be *among* the smallest ones, both of the above arguments qualify as instances of grounding. In that case, however, the very same proposition would have more than one complete ground. Consequently, Bolzano's uniqueness claim, which requires the complete ground of a truth to be uniquely determined, would fail. Bolzano himself seems to be somewhat undecided concerning this claim (cf. section 2), and we will see that giving it up is not entirely implausible, as it also yields tensions with the idea that grounding is preserved under variation.

What we have suggested in this section is reading Bolzano as taking the constraints that he initially introduces as necessary conditions for grounding among conceptual truths and that we discussed in section 3 as jointly sufficient. If successful, this would amount to a kind of definition of grounding.[84] In particular, it would amount to an extensional characterization of grounding for a significant number of cases, namely for all conceptual truths that are logically deducible from their complete grounds.[85] As we have seen in the previous section, however, the constraints Bolzano puts forward as necessary conditions are not entirely unproblematic—especially his constraints to the effect that grounds and consequences must be among the simplest equivalent propositions. Yet, this does not mean that the conditions are implausible as part of a sufficient condition. As we will show in the next section, however, problems occur in connection with Bolzano's conjecture that under suitable conditions grounding is preserved under variation.

4.2 *Grounding Preserved under Variation*

When formulating his suggestion for a sufficient condition for grounding in (B7), Bolzano generalizes his claim to the effect that grounding should be preserved under variation—at least under certain conditions. Let us call a relation that is preserved under variation a *formal relation*. The question whether and to what extent grounding is such a formal relation arises at various places in the *WL*. When Bolzano discusses the relation between deducibility and grounding, he remarks that even though he lacks a proof for this claim, it seems quite obvious to him that if a truth is deducible from its complete ground, grounding is preserved under variation.[86] He also introduces a concept called *formal grounding* (*formale Abfolge*), which is, roughly speaking, a blend of deducibility and grounding.[87] In a case of formal grounding, the conclusion must be deducible from the premises with respect to a given collection of variable ideas, and whenever the variation results in true premises, these must stand in the relation of grounding to the corresponding variant of the conclusion. Formal grounding is an explanatory consequence relation that solely rests on the form of the propositions involved—relative to a given collection of variable ideas.

84Betti ("Explanation in Metaphysics," 295) notes a related point concerning a related passage in Bolzano. Again, our discussion complements hers, as we provide the details of the theory while she places the idea in a broader context of Bolzano's methodology.

85We do not make the claim that the sufficient condition would amount to a definition in the sense of a *conceptual analysis*.

86See *WL*, §200 [II.347–48].

87See *WL*, §162 [II.193].

Whereas Bolzano's definition of formal grounding does not involve any additional restrictions on the admissible substitutions, in (B7) he requires that in order for grounding to be preserved under variation the substitutions must be restricted in such a way that they keep the premises "from becoming redundant."[88] It is not entirely clear what Bolzano precisely means by this. One obvious possibility is to understand him as requiring that the admissible substitutions must be restricted in such a way that they do not yield premises that are more complex than the conclusion or contain deductively superfluous propositions and/or ideas—as is required by the Complexity Constraint as well as Complete and Partial Ground Economy. Since Bolzano regards these conditions as necessary, a case in which one of them is violated cannot possibly constitute a case of grounding.[89] In the following, we will take a closer look at Bolzano's claim that grounding is preserved under variation and investigate where exactly additional restrictions on the admissible substitutions become important.

Let us start by considering an example of an inference rule that Bolzano considers a rule of formal grounding:[90]

*A*s are *B*.
*A*s are *C*.
———————
*A*s are *B* and *C*.

Note that the examples for grounding that we have discussed earlier are instances of this rule. In particular, both of the arguments discussed in the previous subsection (section 4.1) thus fit the form of a rule that Bolzano considers a rule of formal grounding. Recall, however, that both those arguments qualify as cases of grounding only according to our rendering of Bolzano's sufficient condition, because we merely require the respective collection of premises to be *among* the smallest collections of premises (rather than requiring it to be *the* smallest collection). As we said, a consequence is that Bolzano's uniqueness claim (cf. section 2) cannot be upheld. We now see that this is in line with Bolzano suggesting the above rule as a formal grounding rule, of which both of the arguments in question are instances.

The rule under consideration guarantees by its form alone that the premises are never of a higher degree of complexity than the conclusion. No matter what we substitute for the variable ideas, the premises will never contain more occurrences of simple ideas than the conclusion does. Nor is it possible that one of the variants of the premises will contain any occurrence of a simple idea that does not occur in the corresponding variant of the conclusion as well.

[88] *WL*, §221.7 [II.387–88].

[89] For a detailed discussion of Bolzano's sufficient condition and the question to what extent grounding can be considered a formal relation, see Roski, "Bolzano's Notion of Grounding," ch. 5.5, esp. 283–87. The approach taken there slightly differs from the one taken here.

[90] Cf. *WL*, §199 [II.344]; §221.7 [II.388]; and §227 [II.410–11]. As mentioned earlier, Bolzano presents this rule as an illustration for his sufficient condition. Note that because of his views on the form of propositions (cf. section 1 above), Bolzano often presents this argument form under different linguistic guises. However, nothing essential hinges on this for present purposes.

Yet the form of the above rule cannot guarantee that the number of premises remains minimal under variation, nor can it guarantee that the premises are always among the simplest equivalent ones. Without putting further restrictions on the admissible substitutions, we cannot, for instance, rule out a case in which *A* and *B* are replaced by ideas such that the resultant variant of [*A*s are *B*] is logically analytic. An example for such a variant would be [Mortals are mortal].[91] As we have seen, given Complete Ground Economy, such logically analytic truths cannot be part of the complete ground of a given truth, as they constitute redundant premises in logically valid arguments (cf. section 3.2.1). An argument in which such a truth occurs as one of the premises accordingly cannot constitute a case of grounding—despite its being an instance of the above rule.

It is also immediately clear that the form of an argument alone cannot guarantee that in each instance the premises remain among the simplest equivalent propositions. As we have already pointed out, this constraint is problematic in itself and only seems to make sense as a necessary condition for being a fundamental truth (cf. section 3.2.2). In order to see what the requirement amounts to in the context of the sufficient condition, consider the following argument:

The natural numbers are not not well-ordered.
The natural numbers are not not closed under addition.

The natural numbers are not not well-ordered and not not closed under addition.

This argument is also an instance of the grounding rule that we have considered above. Since, according to Bolzano, the proposition [The natural numbers are well-ordered] is logically equivalent to the proposition [The natural numbers are not not well-ordered],[92] it is not the case that each of the premises of this argument is individually among the simplest equivalent ones. The former proposition surely is of a lower degree of complexity than the latter. If it really were required that only propositions that are among the simplest equivalent ones can occur as premises in an instance of grounding, this argument could not constitute a case of grounding. Against the background of the considerations that we raised earlier (cf. section 3.2.2), the question arises whether Bolzano really wants to exclude such cases and whether his respective necessary condition for grounding is thus appropriate at all. Given what he says elsewhere, it seems plausible that he would want to rule out that (iii) is *immediately* grounded in [The natural numbers are well-ordered] and [The natural numbers are closed under addition] but would take (iii) to be immediately grounded in the propositions (i) and (ii). One might thus argue that the restrictions on the admissible substitutions that Bolzano envisages are not meant to exclude the above argument. Given the complexity of the conclusion, the complexity of the premises seems legitimate. It is not clear, however, how this intuition can be made precise.

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[91] This proposition is not only of the form '*A*s are *B*,' but also of the form '*A*s are *A*,' each instance of which is logically analytic. For a discussion of the fact that a proposition or an argument can have different forms in Bolzano's framework see George, "Relevance."

[92] This is strongly suggested by Bolzano's remarks in *WL*, §45 [I.206], and §209 [II.363–64].

While the grounding rule that we have considered so far can guarantee by its form alone that the Complexity Constraint is always fulfilled, it cannot assure that Bolzano's economy requirements remain fulfilled under variation as well. There are, however, also arguments that Bolzano considers cases of grounding, the form of which cannot guarantee that the variants of the premises are always of a lower degree of complexity than the variant of the conclusion. Another logically valid rule that Bolzano discusses in the context of his remarks on grounding is the following:

All *A*s are *B*.
All *B*s are *C*.

All *A*s are *C*.

Bolzano explains that arguments of this form can be considered instances of grounding only if the idea substituted for *B* is not more complex than either the idea substituted for *A* or the idea substituted for *C*.[93] For if the idea substituted for *B* is of a higher degree of complexity than the ideas substituted for *A* or *C* respectively, the Complexity Constraint is violated.[94] In this case, it is the Complexity Constraint that puts restrictions on which substitutions are admissible for grounding to hold under variation.

The above considerations show that Bolzanian grounding is not a formal relation *par excellence*—one that is preserved under *arbitrary* variation. There are cases in which a given truth is deducible from its ground that do not qualify as cases of formal grounding, since some variants are in conflict with conditions Bolzano takes to be necessary for grounding. In order for these conditions to remain fulfilled under variation, the admissible substitutions must be heavily restricted, and while Bolzano hints at some such restrictions, it is not clear how they can be spelled out precisely in completely general terms.

5. CONCLUSION

In this paper we have discussed the part of Bolzano's theory of grounding that involves various principles stating necessary conditions for grounding among truths of the a priori sciences, the so-called conceptual truths. Unlike principles that specify the basic relational properties of grounding, these principles take into account the internal buildup of grounds and consequences. We have argued that they can be seen as motivated by traditional ideas concerning explanation. Bolzano seems to assume that the explanatory order induced by grounding proceeds from the simple to the more complex (Complexity Constraint), and that explanatory arguments exhibit a certain kind of economy: explanations are supposed to make use of as few and as simple truths as possible (Complete and Partial Ground Economy). As we read him, Bolzano at least toys with the idea that each of these principles might not only be necessary for grounding among conceptual truths; they might also be jointly sufficient. Such a sufficient condition for grounding would provide a definitive answer to the question under which

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[93]Cf. *WL*, §221.2 [II.384].
[94]On possible motivations for this constraint; cf. section 3.1 above.

conditions a deductively valid argument is explanatory. Bolzano takes it that from all deductively valid arguments, the explanatory argument stands out due to its simplicity and economy.

We have shown that, given Bolzano's necessary conditions for grounding among conceptual truths, grounding is not preserved under variation in general and thus cannot be considered a formal relation *par excellence.* For grounding to be preserved under variation, the admissible substitutions have to be suitably restricted so that the necessary conditions remain fulfilled. Bolzano's attempt to characterize grounding in terms of various highly general constraints makes grounding a rather exclusive notion: only collections of propositions with certain specific properties are admitted as grounds. This stands in sharp contrast to the idea that there might be cases of grounding that hold only in virtue of the form of the propositions involved, and it seems to rule out the possibility of characterizing grounding partially in terms of formal rules—an idea that is central to many modern approaches to grounding, and that Bolzano himself suggests at some places.

In spite of its internal problems, Bolzano's theory is remarkable. Against the background of his advanced account of the formal consequence relation of deducibility, he was able to make traditional ideas concerning explanations precise to a degree that seems historically unprecedented. That Bolzano arrived at a level of precision at which tensions and problematic aspects of these ideas come to the fore is an important achievement in itself.[95]

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